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EQUIVALENCE OF THREE TOPOLOGIES IN THE SPACES OF LAPLACE-STIELTJES INTEGRALS

For a non-negative nondecreasing unbounded continuous on the right function F and a real-valued function f on $(1,+\infty)$ the integral $I(\sigma)=\int_1^{\infty} f(x)e^{x\sigma}dF(x)$ is called the Laplace-Stieltjes integral. For some class of such integrals three various topologies are introduced and their equivalence is proven.

Key words and phrases: Laplace-Stieltjes integral, space of functions.

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Introduction

Let V be a class of a non-negative nondecreasing unbounded continuous on the right functions F on $(1,+\infty)$. We assume that a real-valued function f on $(1,+\infty)$ is such that the Lebesque-Stieltjes integral $\int_1^A |f(x)|e^{x\sigma}dF(x)$ exists for every $\sigma\in\mathbb{R}$ and $A\in(1,+\infty)$, and the integral

$$I(\sigma)=\int\limits_1^{\infty} f(x)e^{x\sigma}dF(x),\quad \sigma\in\mathbb{R}, \tag{1}$$

is called of Laplace-Stieltjes [1]. We also remark that the Dirichlet series $I(\sigma)=\sum\limits_{n=1}^{\infty} a_n e^{\lambda_n\sigma}$, $1<\lambda_n\uparrow\infty$, can be rewritten in the form (1) with $f(x)=a_n$ for $x=\lambda_n$ and $f(x)=0$ for $x\neq\lambda_n$ and $F(x)=n(x)$, where $n(x)$ is a counting function of (λ_n).

Let

$$M(\sigma)=M(\sigma,I)=\int\limits_1^{\infty} |f(x)|e^{x\sigma}dF(x),\quad \sigma\in\mathbb{R}. \tag{2}$$

It is clear, that if $f(x)\geq 0$ for all $x\geq 0$ then $M(\sigma,I)=I(\sigma)$, and asymptotic properties of integrals of such kind are studied in a monograph [1]. As in [1, p.21] we say that a function $|f|$ has regular

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variation in regard to $F\in V$ if there exist $a\in[0,1)$, $b\in[0,1)$ and $h\in(0,+\infty)>0$ such that for all $x>a$

$$\int\limits_{x-a}^{x+b}|f(t)|dF(t)\geq h|f(x)|. \tag{3}$$

By σ_M we denote an abscissa of the convergence of integral (2), i. e. integral (2) converges for $\sigma<\sigma_M$ and diverges for $\sigma>\sigma_M$. If integral (2) converges for all $\sigma\in\mathbb{R}$ then we put $\sigma_M=+\infty$. It is known [1, p. 21] that if $\ln F(x)=o(x)$ as $x\to+\infty$ and $|f|$ has regular variation in regard to $F\in V$ then $\sigma_M=+\infty$ if and only if

$$\frac{1}{x}\ln\frac{1}{|f(x)|}\to+\infty,\quad x\to+\infty, \tag{4}$$

i. e. $|f(x)|\leq e^{-Kx}$ for every $K>0$ and all $x\geq x_0(K)$. Therefore, in [2] and [3] for a positive continuous on $[0,+\infty)$ function h increases to $+\infty$. By LS_h we denote a class of integrals I such that $|f(x)|\exp\{xh(x)\}\to 0$ as $x\to+\infty$ and define $||I||_h=\sup\{|f(x)|\exp\{xh(x)\}:x\geq 0\}$. For example, it is proven [3] that if $F\in V$ and $\ln F(x)=o(x)$ as $x\to+\infty$ then $(LS_h,||\cdot||_h)$ is a non-uniformly convex Banach space.

In this article we will consider slightly different spaces of integrals (1).

1 Various topologies on $LS(U(F))$

For a fixed function $F\in V$ by $U(F)$ we denote a class of real function on $[1,+\infty)$ such that for every $f_1\in U(F)$ and $f_2\in U(F)$ the functions $|f_1|$, $|f_2|$ and $|f_1-f_2|$ have the regular variation in regard to F, and by $LS(U(F))$ we denote a set of integrals (1) with $f\in U(F)$ and $\sigma_M=+\infty$.

At first we assume that (r_k) is a non-decreasing sequence of positive numbers, $r_k\to+\infty$ with k. If for each $I\in LS(U(F))$

$$||I||_{r_k}=\int\limits_1^{\infty}|f(x)|e^{xr_k}dF(x) \tag{5}$$

then $||I||_{r_k}$ exists for each k and it is easily seen that this is a norm on $LS(U(F))$. It is clear that $||I||_{r_k}\leq||I||_{r_{k+1}}$ for all $k\geq 1$. With these countable norms $||I||_{r_k}$ $(k\geq 1)$ we define (see [4, p.37]) a metric topology on $I\in LS(U(F))$ with metric d:

$$d(I_1,\,I_2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{||I_1-I_2||_{r_k}}{1+||I_1-I_2||_{r_k}},\quad I_1,\,I_2\in LS(F(U)). \tag{6}$$

Since $||I||_{r_k}\leq||I||_{r_{k+1}}$ for all $k\geq 1$, it is clear that the metric topology defined by d is the sup topology which is locally convex (see [4, p. 33-37]).

We remark that there exist integrals (1) such that $M(\sigma,I)\equiv 0$ and $f(x)\not\equiv 0$. Indeed, these is for example if for all $n\in\mathbb{N}$

$$F(x)=\begin{cases}0,\,1\leq x<2,\\ n,\,2n\leq x<2(n+1)\end{cases},\quad f(x)=\begin{cases}\alpha_n>0,\,x=2n-1,\\ 0,\,x\neq 2n-1\end{cases}.$$

However the integral $M(\sigma,I)\in LS(U(F))$ is the additive zero of $LS(U(F))$ if and only if $|f(x)|=0$ for each $x>1$. Indeed, from the regular variation in regard to F we have for $\sigma=0$ and each $x>1$

(see (3))

$$0 = M(\sigma, I) = \int\limits_{1}^{\infty} |f(x)| dF(x) \geq \int\limits_{x-a}^{x+b} |f(t)| dF(t) \geq h|f(x)|,$$

i. e. $f(x) = 0$ for each $x > 1$.

Now for each $I \in LS(U(F))$ let

$$p(I) = \sup_{x>1} |f(x)|^{1/x} \tag{7}$$

and

$$||I||_{q_j} = \sup_{1\leq x\leq q_j} |f(x)|^{1/x}, \tag{8}$$

where (q_j) is a non-decreasing sequence of positive numbers, $1 < q_j \to +\infty$ with j. As (7) is defined by the condition (4). Then the functions $p(f)$ and $||I||_{q_j}$ are paranorms on $LS(U(F))$ (see [5, p.85]).

We define metric topologies on $LS(U(F))$ with the metrics

$$p(I_1,\, I_2) = \sup_{x>1} |f_1(x) - f_2(x)|^{1/x}, \quad I_1,\, I_2 \in LS(U(F)),$$

and

$$s(I_1,\, I_2) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||I_1 - I_2||_{q_j}}{1 + ||I_1 - I_2||_{q_j}}, \quad I_1,\, I_2 \in LS(U(F)).$$

Since $||I||_{q_j} \leq ||I||_{q_{j+1}}$, the topology s is the sup topology which is locally convex.

Theorem 1. *If*

$$\int\limits_{1}^{\infty} e^{-tx} dF(x) \to 0, \quad t \to +\infty, \tag{9}$$

then the three topologies represented by d, p and s are equivalent.

Proof. First we show that the topologies given by d and p are equivalent. Let $I_m \in LS(U(F))$ and $I_m \to I \in LS(U(F))$ as $m \to \infty$ in the paranorm p.

Then if

$$I_m(\sigma) = \int\limits_{1}^{\infty} f_m(x) e^{x\sigma} dF(x), \quad I(\sigma) = \int\limits_{1}^{\infty} f(x) e^{x\sigma} dF(x),$$

we have $|f_m(x) - f(x)| \leq (1/c)^x$ for an arbitrarily large $c > 1$, all $m \geq m_0(c)$ and all $x > 1$. Therefore, due to (9) we have

$$||I_m - I||_{r_k} = \int\limits_{1}^{\infty} |f_m(x) - f(x)| e^{xr_k} dF(x) \leq \int\limits_{1}^{\infty} \exp\{-x(\ln c - r_k)\} dF(x) \to 0$$

as $c \to +\infty$, i. e. $I_m \to I$ as $m \to \infty$ under each norm $||I||_{r_k}$ and $I_m \to I$ as $m \to \infty$ in the metric d.

On the other hand, suppose that $I_m \to I$ as $m \to \infty$ in the metric d, that is under each norm $||I||_{r_k}$. Then for an arbitrarily large $c > 1$, $m \geq m_0(c)$ and all $x > 1$ we have in view of (3) with $|f_m(x) - f(x)|$ instead $f(x)$

$$\frac{1}{c} > \int\limits_{1}^{+\infty} |f_m(t) - f(t)| e^{tr_k} dF(t) \geq \int\limits_{x-a}^{x+b} |f_m(t) - f(t)| e^{tr_k} dF(t) \geq$$

$$\geq e^{(x-a)r_k}\int\limits_{x-a}^{x+b}|f_m(t)-f(t)|dF(t)\geq he^{(x-a)r_k}|f_m(x)-f(x)|, \tag{10}$$

i. e. $|f_m(x)-f(x)|<\dfrac{1}{hc}e^{-(x-a)r_k}\leq\dfrac{1}{c^x}$, provided $r_k\geq\dfrac{1}{x-a}\ln\dfrac{c^{x-1}}{h}$. Thus, for all $m\geq m_0(c)$ all $x>1$ and all $k\geq k_0(c,x)$ we have $|f_m(x)-f(x)|^{1/x}<1/c$, i.e. $\sup\limits_{x\geq 1}|f_m(x)-f(x)|^{1/x}\to 0$ as $m\to\infty$ and $I_m\to I$ in the paranorm p. Hence it follows that the topologies given by d and p are equivalent.

To prove the other part of Theorem 1, let $I_m\to I$ as $m\to\infty$ in the paranorm p. Then $|f_m(x)-f(x)|\leq(1/c)^x$ for an arbitrarily large c, all $m\geq m_0(c)$ and all $x>1$. Therefore,

$$||I_m-I||_{q_j}=\sup_{1\leq x\leq q_j}|f_m(x)-f(x)|^{1/x}\leq 1/c$$

and

$$\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{||I_m-I||_{q_j}}{1+||I_m-I||_{q_j}}\leq\sum_{j=1}^{\infty}\frac{1}{(c+1)2^j}\to 0,\quad c\to\infty,$$

i. e. $I_m\to I$ in the paranorm s.

On the other hand, if $I_m\to I$ in the paranorm s then $||I_m-I||_{q_j}\to 0$ as $m\to\infty$ for each q_j and, thus, $|f_m(x)-f(x)|^{1/x}\leq 1/c$ for an arbitrarily large c, all $m\geq m_0(c)$, all $x\in(1,\,q_j]$ and all q_j, that is for all $x>1$. Hence $|f_m(x)-f(x)|^{1/x}\to 0$ as $m\to\infty$ for all $x>1$ and, therefore, $I_m\to I$ in the paranorm p. Thus, the topologies given by p and s are equivalent. Theorem 1 is proved. □

The following theorem establishes a connection between the convergence under $p(I)$ and the convergence on every finite interval.

Theorem 2. *If F satisfies* (9) *and a sequence $(I_m)\subset LS(U(F))$ converges to $I\in LS(U(F))$ under $p(I)$ then $I_m\to I$ uniformly converges on every finite interval.*

Proof. Let $p(I_m-I)\to 0$ as $m\to\infty$. Then for a given $\varepsilon>0$, there exists an $m_0=m_0(\varepsilon)$ such that $|f_m(x)-f(x)|\leq\varepsilon^x$ for $m\geq m_0$, or $|f_m(x)-f(x)|\leq\varepsilon^{-nx}$, where $n\in\mathbb{N}$ is arbitrarily large. Therefore, for $m\geq m_0$ and $\sigma\in[\sigma_1,\,\sigma_2]$ due to (9) we have

$$|I_m(\sigma)-I(\sigma)|<\int\limits_1^{+\infty}e^{x(\sigma_2-n)}dF(x)\to 0,\quad n\to\infty.$$

Theorem 2 is proved. □

Remark 1. An opposite statement to Theorem 2 does not hold. Indeed, let for every $m\in\mathbb{Z}_+$ and $n\in\mathbb{N}$

$$F(x)=\begin{cases}0,\ 1\leq x<2,\\ n,\ 2n\leq x<2(n+1)\end{cases},\quad f_m(x)=\begin{cases}\alpha_{m,n}>0,\ x=2n-1,\\ 0,\ x\neq 2n-1\end{cases}.$$

Then for all $m\in\mathbb{Z}_+$

$$I_m(\sigma)=\int\limits_0^{\infty}f_m(x)e^{x\sigma}dF(x)=\sum_n f_m(2n)e^{2n\sigma}=0,$$

i. e. $I_m(\sigma)\to I_0(\sigma)$ for all $\sigma\in[\sigma_1,\,\sigma_2]$. On the other hand,

$$p(I_m,I_0)=\sup\left\{|f_m(x)-f_0(x)|^{1/x}\,:\,x>1\right\}\geq$$
$$\geq|f_m(3)-f_0(3)|^{1/3}=|\alpha_{m,3}-\alpha_{0,3}|^{1/3}\geq h_1>0$$

provided $\alpha_{m,3}-\alpha_{0,3}\geq\eta_1>0$ for all $m\in\mathbb{N}$.

2 COMPLETENESS.

Now we will show the completeness of the space $LS(U(F))$ under various topologies established above. Note that it is sufficient to establish the completeness under one of them, due to Theorem 1.

Theorem 3. *If F satiesfies* (10) *then the space* $(LS(U(F)),s)$ *is complete.*

Proof. Let (I_ν) be a s-Cauchy sequence on $LS(U(F))$. Then for a given $\varepsilon\in(0,\,1)$ there exists a $Q=Q(\varepsilon)$ such that for all $\nu\geq Q$ and $n\geq Q$

$$\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{||I_\nu-I_n||_{q_j}}{1+||I_\nu-I_n||_{q_j}}<\varepsilon,$$

whence

$$|f_\nu(x)-f_n(x)|^{1/x}<\varepsilon_1,\quad \nu\geq Q,n\geq Q,1\leq x\leq q_j,$$

where $\varepsilon_1=\varepsilon_1(\varepsilon)\to0$ as $\varepsilon\to0$. Hence the sequence (f_ν) in p tends to $f(x)$ for each $x>1$. Since $\dfrac{1}{x}\ln\dfrac{1}{|f_\nu(x)|}\to+\infty$ as $x\to+\infty$, we have $f_\nu(x)\leq\varepsilon_1^x$ for $x\geq x_0(\varepsilon_1)$ and, therefore,

$$|f(x)|\leq|f_\nu(x)-f(x)|+f|_\nu(x)|\leq2\varepsilon_1^x,\quad x\geq x_0(\varepsilon_1).$$

Thus, (4) holds for f and $I^*(\sigma)=\int\limits_1^\infty|f(x)|e^{x\sigma}dF(x)$ has the abscissa of the convergence $\sigma_M=+\infty$.

Since all $f_\nu\in U$, the function $f\in U$. Indeed, if $|f|$ does not have a regular variation in regard to $F\in V$ then for all $a\geq0$, all $b\geq0$ and all $h>0$ there exists $x*>1+a$ such that

$$\int\limits_{x^*-a}^{x^*+b}|f(t)|dF(t)<h|f(x^*)|. \tag{11}$$

Clearly, $|f(x^*)|>0$ and, therefore, $f_\nu(x^*)>0$ for $\nu\geq\nu_0$. Since f_ν has regular variation in regard to $F\in V$, there exist $a\in[0,1)$, $b\in[0,1)$ and $h_1>0$ such that

$$\int\limits_{x^*-a}^{x^*+b}|f_\nu(t)|dF(t)\geq h_1|f_\nu(x^*)|. \tag{12}$$

But $f_\nu(x^*)\to f(x^*)$ as $\nu\to\infty$. Therefore, from (11) and (12) we obtain for $\nu\to\infty$

$$h_1|f(x^*)|+o(1)=h_1|f_\nu(x^*)|\leq\int\limits_{x^*-a}^{x^*+b}|f_\nu(t)|dF(t)$$
$$=\int\limits_{x^*-a}^{x^*+b}|f(t)|dF(t)+o(1)<h|f(x^*)|+o(1).$$

This is impossible of the arbitrariness of h. Thus, $I \in LS(U(F))$.

Finally,

$$||I_\nu - I||_{q_j} = \sup_{1<x\le q_j} |f_\nu(x) - f(x)|^{1/x} < \varepsilon$$

for all $\nu \ge Q$. Hence

$$s(I_\nu,\, I) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||I_\nu - I||_{q_j}}{1+||I_\nu - I||_{q_j}} < \frac{\varepsilon}{1+\varepsilon}, \quad \nu \ge Q,$$

i. e. $(LS(U(F)), s)$ is complete. Theorem 3 is proved. □

Corollary 1. *If F satisfies* (9) *then the spaces $(LS(U(F)), s)$, $(LS(U(F)), p)$, $(LS(U(F)), d)$ and the space $LS(U(F))$ endowed with the compact open topology (as in Theorem 2) are Frechet spaces and, thus, are barrelled spaces.*

Theorem 4. *If F satiesfies* (9) *then the space $(LS(U(F)), p)$ is a Montel space (see [6, p.32]).*

Proof. Let X be an arbitrary uniformly bounded subset of $(LS(U(F)), p)$, i. e. there exists $C \in (1, +\infty)$ such that $p(I) \le C$ for all $I \in X$. For $I \in X$ as from (7) we have $|f(x)| \le C^x$ for all $x > 1$ and, as above, $|f(x)| \le e^{-nx}$ for each $n > 1$ and all $x \ge x_0 = x_0(n)$. Therefore, for $\sigma \in D := [\sigma_1,\, \sigma_2]$ we have

$$|I'(\sigma)| \le \int_1^{+\infty} x|f(x)|e^{x\sigma_2}dF(x) \le \left(\int_1^{x_0} + \int_{x_0}^{\infty}\right) |f(x)|e^{x(\sigma_2+1)}dF(x) \le$$

$$\le \max\{C^x e^{x(\sigma_2+1)} :\, 1 \le x \le x_0\} \int_1^{x_0} dF(x) + \int_{x_0}^{\infty} e^{-(n-\sigma_2-1)x} dF(x).$$

Hence from the arbitrariness of n and (9) we have $|I'(\sigma)| \le C_D$ for all $I \in X$ and $\sigma \in D$, where C_D is a constant depending on D, and for $\sigma', \sigma'' \in D$, $\sigma' < \sigma''$, we obtain for some $\xi \in [\sigma', \sigma'']$

$$|I(\sigma'') - I(\sigma')| = \int_1^{+\infty} x|f(x)|e^{x\xi}dF(x) \le C_D^*(\sigma'' - \sigma')$$

for all $I \in X$, i. e. X is equi-continuos. Now by a well-known argument we can select a subsequence of X which converges uniformly on D to a function I.From the arbitrariness of D and the completeness of $(LS(U(F)), p)$ Theorem 4 is proved. □

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Для невід'ємної, неспадної, необмеженої справа, неперервної функції F та дійснозначної функції f, заданої на $(1,+\infty)$, інтеграл $I(\sigma)=\int_1^{\infty}f(x)e^{x\sigma}dF(x)$ називається інтегралом Лапласа-Стілтьєса. Для певного класу таких інтегралів введено три топології та доведено їхню еквівалентність.