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EXTENSION OF THE CLASS OF INITIAL DATA OF THE CAUCHY PROBLEM FOR THE ISOTROPIC SUPERDIFFUSION EQUATION

We consider the Cauchy problem for the isotropic superdiffusion equation with the Riesz fractional differentiation operator of order $\alpha \in (0;2)$, which generalizes the classical heat conduction equation. Such models arise in the description of anomalous transport of energy and mass in fractal and porous media, plasma, and other complex structures exhibiting superdiffusive behavior. The Riesz operator is the generator of symmetric α-stable Levy processes; therefore, the solution of the Cauchy problem may be interpreted as the probability density of the corresponding stochastic process.

We prove the existence of a classical bounded smooth solution, even when the initial data contain a finite number of integrable discontinuities of the second kind.

Key words and phrases: Isotropic superdiffusion equation; Riesz operator; fractional Laplacian; Cauchy problem; symmetric α-stable Levy process.

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Introduction

Consider the equation

$$\partial_t u(t;x) + a(-\Delta)_x^{\alpha/2} u(t;x) = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{1}$$

where $a > 0$, u is the unknown function, ∂_t denotes the partial derivative with respect to the variable t, and $A_x^\alpha \equiv (-\Delta)_x^{\alpha/2}$ is the Riesz fractional differentiation operator (i.e., the fractional Laplacian of order $\alpha \in (0;2)$) acting in the variable $x \in \mathbb{R}^n$ according to the rule

$$(A_x^\alpha f)(x) = c(\alpha) \begin{cases} \int\limits_{\mathbb{R}^n} \frac{f(x)-f(x+y)+[\alpha]\big(y,\mathrm{grad} f(x)\big)}{|y|^{n+\alpha}} dy, & \alpha \neq 1, \\ \lim\limits_{\varepsilon \to 0} \int\limits_{|y|>\varepsilon} \frac{f(x)-f(x+y)}{|y|^{n+\alpha}} dy, & \alpha = 1. \end{cases} \tag{2}$$

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Here $[\cdot]$ denotes the integer part, and

$$c(\alpha) = \begin{cases} \frac{2^\alpha \Gamma(1+\alpha/2)\Gamma((n+\alpha)/2)}{\pi^{n/2}\Gamma(\alpha/2)\Gamma(1-\alpha/2)}, & 0 < \alpha < 1, \\ \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}, & \alpha = 1, \\ \frac{\alpha\Gamma((3-\alpha)/2)\Gamma((n+\alpha)/2)}{\pi^{(n+1)/2}\Gamma(2-\alpha)}, & 1 < \alpha < 2, \end{cases}$$

is the corresponding coefficient, where $\Gamma(\cdot)$ denotes Euler's gamma function. On sufficiently smooth functions $f(\cdot)$, the action of this operator is also given by [12]

$$(A_x^\alpha f)(x) = F^{-1}_{\xi\to x}\big[|\xi|^\alpha F_{x\to\xi}[f](\xi)\big](x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

where $F_{\xi\to x}$ denotes the Fourier transform with respect to ξ.

Note that for $\alpha = 2$, equation (1) reduces to the classical heat equation. This equation has found wide applications in many areas of science and engineering [13, 3, 15, 11, 16]. It describes the propagation of resonant photons in plasma and diffusion processes in fractal media [14, c. 251], which is why equation (1) is often referred to as the *isotropic superdiffusion equation*. Equation (1) is also used to model local vortices of Riesz-type gravitational fields generated by moving objects [8, 10, 9].

An important example motivating the study of the isotropic superdiffusion equation is given in [1, c. 2], where a probabilistic model of a random jump-type walk of a particle X on $\mathbb{R}$is proposed, and it is shown that the probability $u(t;x)$ of finding X at point x at time t is a solution of equation (1) (for the case $n = 1$). A typical example of such motion is the darting of a hungry shark or the flight of a swift hunting insects.

The fundamental solution

$$G_\alpha(t;x) = F^{-1}_{\xi\to x}[e^{-ta|\xi|^\alpha}](t;x), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{3}$$

of the Cauchy problem for equation (1) is the probability density of a symmetric α-stable Levy stochastic process [17]. The Cauchy problem for equation (1) and the function $G_\alpha(t;x)$ has been studied in many papers (see the survey in [2, 4]). Various methods have been developed to analyze the properties of $G_\alpha(t;x)$, and numerous results have been obtained on the well-posedness of the Cauchy problem in different functional classes for continuous bounded initial data. Typical properties of classical solutions of equation (1) and some of its generalizations have also been established.

In this paper, the well-posedness of the Cauchy problem for equation (1) is established in the case when the initial functions on $\mathbb{R}^n$ are almost everywhere continuous, locally integrable, and may have a finite number of discontinuities of the second kind. In this way, the known results for bounded continuous initial data are generalized.

1 Preliminary information. Problem statement

Let $\mathbb{N}_m = \{1, \dots, m\}$; let $\mathbb{R}^n$ denote the n-dimensional Euclidean space equipped with the scalar product $(\cdot,\cdot)$ and the norm $|x| = (x,x)^{1/2}$, and let $\mathbb{Z}^n_+$ denote the set of all multi-indices of dimension n.

In the case $\alpha = 1$, one can compute the Fourier transform in the right-hand side of equality (3) and obtain an explicit expression for the fundamental solution G_1:

$$G_1(t;x) = \frac{\Gamma(n/2)}{(2\pi)^{\frac{n}{2}}} \frac{at}{\big((at)^2 + x^2\big)^{\frac{n+1}{2}}}, \quad t > 0, \quad x \in \mathbb{R}^n. \tag{4}$$

This fact confirms that, unlike the classical heat equation (the case $\alpha = 2$), the fundamental solution of the Cauchy problem for equation (1) exhibits a power-law behavior at infinity, rather than exponential decay.

For the derivatives of the function G_α, the following estimates hold (see [5, 7]):

$$|\partial_x^k G_\alpha(t;x)| \le c_{1,k} t \left(t^{1/\alpha} + |x|\right)^{-n-|k|-\alpha}; \tag{5}$$

$$|\partial_t \partial_x^k G_\alpha(t;x)| \le c_{2,k} \left(t^{1/\alpha} + |x|\right)^{-n-|k|-\alpha}, \tag{6}$$

where $c_{1,k}$ and $c_{2,k}$ are positive constants depending only on $k \in \mathbb{Z}_+^n$.

The fundamental solution of the Cauchy problem for equation (1) possesses the following properties:

1°. $G_\alpha(t;\cdot)$ is a radial function;

2°. $G_\alpha(t;\cdot)$ is a positive function;

3°. $G_\alpha(t;\cdot)$ is unimodal for every fixed t;

4°. the following equality holds:

$$\int_{\mathbb{R}} G_\alpha(t;x-\zeta)dx = 1, \quad t > 0, \quad \zeta \in \mathbb{R}^n. \tag{7}$$

Note that property 1° follows from the corresponding property of the Fourier transform of a radial function; properties 2° and 3° are well-known results from the theory of symmetric stable stochastic processes (see details in [17]). Equality (7) follows directly from the Fourier inversion formula.

Definition 1. *We call the domain of the operator A_x^α, denoted by $\mathcal{D}(A_x^\alpha)$, the set of all functions $f(\cdot)$ defined on $\mathbb{R}^n$ for which the right-hand side of representation* (2) *is finite for the corresponding value of α.*

Next, consider a function $f(\cdot)$ that is continuous on the set $\mathbb{R}^n \setminus \{x_j\}_{j=1}^m$, bounded at infinity, and such that in sufficiently small neighborhoods of each point x_j the following estimates

$$|f(x)| \le \frac{c}{|x-x_j|^{\gamma_j}}, \quad 0 \le \gamma_j < n, \quad j \in \mathbb{N}_m, \tag{8}$$

hold. Set

$$\varepsilon_0 := \min_{j \neq l} \frac{|x_j - x_l|}{2}, \quad \hat{\varepsilon}_0 := \min\{\varepsilon_0; 1\}, \quad \mathbb{U}_j := \{x \in \mathbb{R}^n : |x - x_j| < \hat{\varepsilon}_0\}, \quad \mathbb{U} := \bigcup_{j=1}^m \mathbb{U}_j.$$

Note that the function $f(\cdot)$ is bounded on $\mathbb{R}^n \setminus \mathbb{U}$ and integrable on every compact set $\mathbb{K} \subset \mathbb{R}^n$.

For equation (1), we consider the Cauchy problem with the initial condition

$$u(t;\cdot)_{|t=0} = f(\cdot). \tag{9}$$

Definition 2. *A function $u(t;x)$ is called a solution of the Cauchy problem* (1), (9) *on the set $\Pi_+ = \{(t;x) : t > 0, x \in \mathbb{R}^n\}$ if it is differentiable in t on this set and satisfies $u(t;\cdot) \in \mathcal{D}(A_x^\alpha)$ for all $t > 0$. Moreover, u satisfies equation* (1) *on Π_+ in the classical sense and attains the initial condition* (9) *in the sense of pointwise limit*

$$u(t;x) \underset{t\to+0}{\to} f(x), \quad x \in \mathbb{R}^n \setminus \{x_j\}_{j=1}^m. \tag{10}$$

The problem consists in finding a solution of the Cauchy problem (1), (9) and proving its uniqueness.

2 THE MAIN RESULT

We will need the following auxiliary statement.

Lemma 1. *Let*

$$u(t;x) = (f * G_\alpha)(t;x), \quad (t;x) \in \Pi_+. \tag{11}$$

Then the function u satisfies:

1) for each fixed $t > 0$, $u(t;\cdot)$ is infinitely differentiable in x on $\mathbb{R}^n$ and all its derivatives are bounded;

2) for each fixed $x \in \mathbb{R}^n$, the function $u(t;\cdot)$ is differentiable in t on $(0;+\infty)$;

3) for each fixed $t \in (0;+\infty)$, $u(t;\cdot)$ belongs to $\mathcal{D}(A_x^\alpha)$.

Moreover, for all $(t;x) \in \Pi_+$ the equalities

$$\partial_x^k u(t;x) = (f * \partial_x^k G_\alpha)(t;x), \quad \partial_t u(t;x) = (f * \partial_t G_\alpha)(t;x),$$
$$A_x^\alpha u(t;x) = (f * A_x^\alpha G_\alpha)(t;x) \tag{12}$$

hold.

Proof. Observe that

$$(f * G_\alpha)(t;x) = \int_{\mathbb{R}^n} f(y) G_\alpha(t;x-y)dy, \quad (t;x) \in \Pi_+.$$

We first verify the existence of the first-order partial derivatives $\partial_x^1 u(t;\cdot)$ on $\mathbb{R}^n$ for every fixed $t > 0$. Fix an arbitrary point $x_0 \in \mathbb{R}^n$ and consider the ball

$$\mathbb{K}_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \le r\}, \quad r > 0.$$

Let

$$\Phi_t(x;\xi) = ct\left(t^{1/\alpha} + |x-\xi|\right)^{-n-1-\alpha}.$$

Clearly, for all $x \in \mathbb{K}_1(x_0)$ we have

$$\Phi_t(x;\xi) \le \begin{cases} ct^{\frac{n+1}{\alpha}}, & \xi \in \mathbb{K}_2(x_0), \\ ct\left(t^{1/\alpha} + (|x_0-\xi|-1)\right)^{-n-1-\alpha}, & \xi \in \mathbb{R}^n \setminus \mathbb{K}_2(x_0). \end{cases}$$

In view of this estimate and the bound

$$f(\xi) \le \begin{cases} c|\xi - x_j|^{-\gamma_j}, & \xi \in \mathbb{U}_j, \quad j \in \mathbb{N}_m, \\ c, & \xi \in \mathbb{R}^n \setminus \mathbb{U}, \end{cases}$$

we conclude that there exists a positive function $\hat{\Phi}_t(\cdot) \in L_1(\mathbb{R}^n)$, $t > 0$, such that for all $x \in \mathbb{K}_1(x_0)$ and $\xi \in \mathbb{R}^n$ the inequality

$$|f(\xi)G_\alpha(t;x-\xi)| \le \hat{\Phi}_t(\xi), \quad t > 0,$$

holds.

Then, by the standard result on differentiation of an integral with respect to a parameter under the Lebesgue dominated convergence condition [6, p. 368], we obtain

$$\partial_x^1 \int_{\mathbb{R}} G_\alpha(t;x-\xi)\varphi(\xi)d\xi = \int_{\mathbb{R}} \partial_x^1 G_\alpha(t;x-\xi)\varphi(\xi)d\xi \quad (\forall x : |x - x_0| < 1).$$

Hence, by the arbitrariness of the point x_0, we conclude that the solution $u(t;x)$ is differentiable at every $x \in \mathbb{R}^n$ for $t > 0$, and that

$$\partial_x^1 u(t;x) = \int\limits_{\mathbb{R}} \partial_x^1 G_\alpha(t;x-\xi)\varphi(\xi)d\xi.$$

In a similar way, one verifies the existence of the remaining derivatives $\partial_x^k u(t;\cdot)$ on $\mathbb{R}^n$ for $t > 0$, as well as the validity of the first identity in (12).

Next, we justify the boundedness of $\partial_x^k u(t;\cdot)$ for every fixed $t > 0$. Using estimates (5) and (8), we obtain

$$\forall k \in \mathbb{Z}_+^n \ \forall (t;x) \in \Pi_+ : \quad \Big|\int\limits_{\mathbb{R}^n} f(y)\partial_x^k G_\alpha(t;x-y)dy\Big| \leq$$

$$\leq \sum_{j=1}^{m} \int\limits_{|y-x_j|<1} |f(y)||\partial_{x-y}^k G_\alpha(t;x-y)|dy + \int\limits_{\mathbb{R}^n\setminus\mathbb{U}} |f(y)||\partial_{x-y}^k G_\alpha(t;x-y)|dy \leq$$

$$\leq \frac{c_k}{t^{n+|k|}} \sum_{j=1}^{m} \int\limits_{|y-x_j|<1} \frac{dy}{|y-x_j|^{\gamma_j}} + \sum_{j=1}^{m} \hat{\varepsilon}_0^{-\gamma_j} \int\limits_{\mathbb{R}^n} \frac{c_k t dy}{\big(t^{1/\alpha}+|x-y|\big)^{n+|k|+\alpha}} =$$

$$= \frac{\tilde{c}_k}{t^{n+|k|}} + \sum_{j=1}^{m} \frac{c_k}{\hat{\varepsilon}_0^{\gamma_j} t^{|k|/\alpha}} \int\limits_{\mathbb{R}^n} \frac{dz}{(1+|z|)^{n+|k|+\alpha}} \equiv \frac{\tilde{c}_k}{t^{n+|k|}} + \frac{\tilde{b}_k}{t^{|k|/\alpha}}$$

(where the positive constants $\tilde{c}_k$ and $\tilde{b}_k$ depend only on k). These estimates guarantee the boundedness of the derivatives $\partial_x^k u$ on every set $[\delta;+\infty) \times \mathbb{R}^n$, $\delta > 0$.

In a similar way, using estimates (6) and (8), we verify that the function u is differentiable with respect to t, and the second equality in (12) holds.

Let us now proceed to establishing the third equality in (12).

First, consider the case $\alpha < 1$. According to the definition of the operator A_x^α, for $(t;x) \in \Pi_+$ we have

$$A_x^\alpha u(t;x) = c(\alpha) \int\limits_{\mathbb{R}^n} \Big(|y|^{-(n+\alpha)} \int\limits_{\mathbb{R}^n} \Big(G_\alpha(t;x-\xi) - G_\alpha(t;x-\xi+y)\Big) f(\xi)d\xi\Big)dy. \tag{13}$$

Hence, we see that the justification of this equality reduces to proving the possibility of changing the order of integration in (13). According to the corresponding statement of Fubini's theorem, it is sufficient to establish the absolute convergence of the iterated integral in (13). Let us prove this convergence for $(t;x) \in \Pi_+$:

$$\Big|\int\limits_{\mathbb{R}^n} \Big(|y|^{-(n+\alpha)} \int\limits_{\mathbb{R}^n} \Big(G_\alpha(t;x-\xi) - G_\alpha(t;x-\xi+y)\Big) f(\xi)d\xi\Big)dy\Big| \leq$$

$$\leq \sum_{j=1}^{m} \int\limits_{\mathbb{R}^n} \Big(|y|^{-(n+\alpha)} \int\limits_{|\xi-x_j|<1} |G_\alpha(t;x-\xi) - G_\alpha(t;x-\xi+y)||f(\xi)|d\xi\Big)dy +$$

$$+ \int\limits_{\mathbb{R}^n} \Big(|y|^{-(n+\alpha)} \int\limits_{\mathbb{R}^n\setminus\mathbb{U}} |G_\alpha(t;x-\xi) - G_\alpha(t;x-\xi+y)||f(\xi)|d\xi\Big)dy \equiv \mathfrak{I}_1(t;x) + \mathfrak{I}_2(t;x).$$

Using the Lagrange theorem on finite increments together with estimates (5) and (8), we obtain:

$$\mathfrak{I}_1(t;x) \leq c\sum_{j=1}^{m}\Big(\sum_{l=1}^{n}\int\limits_{|y|<1}\int\limits_{|\xi-x_j|<1}\frac{|\partial_{z_l}G_\alpha(t;x-\xi+y'+\theta_l y_l)|}{|\xi-x_j|^{\gamma_j}|y|^{n+\alpha-1}}d\xi dy+$$

$$+\int\limits_{|y|\geq 1}\frac{dy}{|y|^{n+\alpha}}\int\limits_{|\xi-x_j|<1}\frac{|G_\alpha(t;x-\xi)|d\xi}{|\xi-x_j|^{\gamma_j}}+\int\limits_{|y|\geq 1}\int\limits_{|\xi-x_j|<1}\frac{|G_\alpha(t;x-\xi+y)|}{|\xi-x_j|^{\gamma_j}}d\xi dy\Big)\leq$$

$$\leq c\sum_{j=1}^{m}\Big(\sum_{l=1}^{n}t^{-\frac{n+1}{\alpha}}\int\limits_{|y|<1}\frac{dy}{|y|^{n+\alpha-1}}\int\limits_{|\xi-x_j|<1}\frac{d\xi}{|\xi-x_j|^{\gamma_j}}+t^{-\frac{n}{\alpha}}\int\limits_{|y|\geq 1}\frac{dy}{|y|^{n+\alpha}}\int\limits_{|\xi-x_j|<1}\frac{d\xi}{|\xi-x_j|^{\gamma_j}}+$$

$$+\int\limits_{\mathbb{R}^n}G_\alpha(t;z)dz\int\limits_{|\xi-x_j|<1}\frac{d\xi}{|\xi-x_j|^{\gamma_j}}\Big)\leq c(1+t^{-\frac{n}{\alpha}}+t^{-\frac{n+1}{\alpha}}),\quad \theta_l\in(0;1),(t;x)\in\Pi_+;$$

$$\mathfrak{I}_2(t;x)\leq c\Big(\sum_{l=1}^{n}\int\limits_{|y|<1}\int\limits_{\mathbb{R}^n\setminus\mathbb{U}}\frac{|\partial_{z_l}G_\alpha(t;x-\xi+y'+\theta_l y_l)|}{|y|^{n+\alpha-1}}d\xi dy+$$

$$+\int\limits_{|y|\geq 1}\Big(|y|^{-(n+\alpha)}\int\limits_{\mathbb{R}^n\setminus\mathbb{U}}(G_\alpha(t;x-\xi)+G_\alpha(t;x-\xi+y))d\xi\Big)dy\leq$$

$$\leq c\Big(\sum_{l=1}^{n}t^{-\frac{1}{\alpha}}\int\limits_{|y|<1}\frac{dy}{|y|^{n+\alpha-1}}\int\limits_{\mathbb{R}^n}\frac{dz}{(1+|z|)^{n+\alpha+1}}+2\int\limits_{\mathbb{R}^n}G_\alpha(t;z)dz\int\limits_{|y|\geq 1}\frac{dy}{|y|^{n+\alpha}}\Big)\leq$$

$$\leq c(1+t^{-\frac{1}{\alpha}}),\quad \theta_l\in(0;1),\quad (t;x)\in\Pi_+.$$

From here we already obtain the required absolute convergence on the set Π_+ of the repeated integral on the right-hand side of equality (13).

The case $\alpha>1$ is treated analogously.

Let us now consider the case $\alpha=1$. To avoid cumbersome arguments, we assume $n=1$. The general case of arbitrary n is handled in a similar manner.

According to the definition of the operator A_x^1, for $(t;x)\in\Pi_+$ we have

$$A_x^1u(t;x)=c(1)\lim_{\varepsilon\to 0}\int\limits_{|y|>\varepsilon}\Big(|y|^{-2}\int\limits_{\mathbb{R}}\Big(G_1(t;x-\xi)-G_1(t;x-\xi+y)\Big)f(\xi)d\xi\Big)dy. \tag{14}$$

Since the repeated integral on the right-hand side of (15) is absolutely convergent on Π_+ for every $\varepsilon>0$–which is not difficult to verify, taking into account estimates (5) and (8)–the previous equality can be rewritten in the following form:

$$A_x^1u(t;x)=c(1)\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}}\Big(f(\xi)\int\limits_{|y|>\varepsilon}\frac{G_1(t;x-\xi)-G_1(t;x-\xi+y)}{|y|^2}dy\Big)d\xi\equiv$$

$$\equiv c(1)\Big(\int\limits_{\mathbb{R}}\Big(f(\xi)\int\limits_{|y|\geq 1}\frac{G_1(t;x-\xi)-G_1(t;x-\xi+y)}{|y|^2}dy\Big)d\xi+$$

$$+\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}}\Big(f(\xi)\int\limits_{1>|y|>\varepsilon}\frac{G_1(t;x-\xi)-G_1(t;x-\xi+y)}{|y|^2}dy\Big)d\xi\Big).$$

Therefore, the proof comes down to substantiating the correctness of the equality

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}}\Big(f(\xi)I_\varepsilon(t;x-\xi)\Big)d\xi=\int\limits_{\mathbb{R}}\Big(f(\xi)\lim_{\varepsilon\to 0}I_\varepsilon(t;x-\xi)\Big)d\xi,$$

where

$$I_\varepsilon(t;\zeta)=\int\limits_{1>|y|>\varepsilon}\frac{G_1(t;\zeta)-G_1(t;\zeta+y)}{|y|^2}dy.$$

According to the well–known statement on passing to the limit under the integral sign [6], it is sufficient to establish the existence of a nonnegative function $g(t;\zeta)$, independent of ε, such that:

I) $|I_\varepsilon(t;\zeta)|\le g(t;\zeta),\quad \varepsilon\in(0;1),\ \zeta\in\mathbb{R},\ t>0;$

II) $\int\limits_{\mathbb{R}}|f(\xi)|g(t;x-\xi)d\xi<+\infty,\quad x\in\mathbb{R},\ t>0.$

First, we compute the value of the integral I_ε. Using the structure (4) of the function G_1, we transform the integrand as follows:

$$\frac{G_1(t;\zeta)-G_1(t;\zeta+y)}{|y|^2}=\frac{1}{\sqrt{2}y^2}\Big(\frac{\tilde a}{\tilde a^2+\zeta^2}-\frac{\tilde a}{\tilde a^2+(\zeta+y)^2}\Big)=$$

$$=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\frac{2\zeta y+y^2}{y^2(\tilde a^2+(\zeta+y)^2)}=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\Big(\frac{2\zeta}{y(\tilde a^2+(\zeta+y)^2)}+\frac{1}{\tilde a^2+(\zeta+y)^2}\Big)=$$

$$=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\left(\frac{2\zeta}{\tilde a^2+\zeta^2}\Big(\frac{1}{y}-\frac{y+2\zeta}{\tilde a^2+(\zeta+y)^2}\Big)+\frac{1}{\tilde a^2+(\zeta+y)^2}\right)$$

(here $\tilde a=at$). Then

$$I_\varepsilon(t;\zeta)=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\left(\frac{2\zeta}{\tilde a^2+\zeta^2}\Big(\int\limits_{1>|y|>\varepsilon}\frac{dy}{y}-\int\limits_{1>|y|>\varepsilon}\frac{(\zeta+y)dy}{\tilde a^2+(\zeta+y)^2}-\right.$$

$$\left.-\zeta\int\limits_{1>|y|>\varepsilon}\frac{dy}{\tilde a^2+(\zeta+y)^2}\Big)+\int\limits_{1>|y|>\varepsilon}\frac{dy}{\tilde a^2+(\zeta+y)^2}\right)=$$

$$=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\left(\Big(1-\frac{2\zeta^2}{\tilde a^2+\zeta^2}\Big)\int\limits_{1>|y|>\varepsilon}\frac{dy}{\tilde a^2+(\zeta+y)^2}-\frac{\zeta}{\tilde a^2+\zeta^2}\int\limits_{1>|y|>\varepsilon}\frac{d(\tilde a^2+(\zeta+y)^2)}{\tilde a^2+(\zeta+y)^2}\right)=$$

$$=\frac{\tilde a}{\sqrt{2}(\tilde a^2+\zeta^2)}\left(\frac{\tilde a^2-\zeta^2}{\tilde a(\tilde a^2+\zeta^2)}\Big(\arctan\frac{\zeta-\varepsilon}{\tilde a}-\arctan\frac{\zeta+\varepsilon}{\tilde a}+\arctan\frac{\zeta+1}{\tilde a}-\arctan\frac{\zeta-1}{\tilde a}\Big)-\right.$$

$$\left.-\frac{\zeta}{\tilde a^2+\zeta^2}\ln\left(\frac{\tilde a^2+(\zeta-\varepsilon)^2}{\tilde a^2+(\zeta-1)^2}\cdot\frac{\tilde a^2+(\zeta+1)^2}{\tilde a^2+(\zeta+\varepsilon)^2}\right)\right),\quad \varepsilon\in(0;1),\quad t>0,\quad \zeta\in\mathbb{R}.$$

For convenience, introduce the notation

$$\Phi_{1,\varepsilon}(\zeta)=\frac{\tilde a^2+(\zeta-\varepsilon)^2}{\tilde a^2+(\zeta-1)^2},\quad \Phi_{2,\varepsilon}(\zeta)=\frac{\tilde a^2+(\zeta+1)^2}{\tilde a^2+(\zeta+\varepsilon)^2}.$$

Using standard tools of differential calculus, one verifies the validity of the estimates

$$\frac{\tilde a^2}{\tilde a^2+(1+\sqrt{1+\tilde a^2})^2}\le\Phi_{1,\varepsilon}(\zeta)\le\frac{4+3\tilde a^2}{\tilde a^2},\quad \varepsilon\in(0;1),\quad \tilde a\ne 0,\quad \zeta\in\mathbb{R}.$$

Taking into account the equality

$$\Phi_{2,\varepsilon}(\zeta) = \frac{1}{\Phi_{1,\varepsilon}(-\zeta)},$$

and also the strict monotonicity of $\ln(\cdot)$ together with the boundedness of $\arctan(\cdot)$, we obtain the following estimate:

$$|I_\varepsilon(t;\zeta)| \le \frac{b(t)}{(at)^2+\zeta^2}, \quad \varepsilon \in (0;1), \quad \zeta \in \mathbb{R}, \quad t>0$$

(where the positive quantity $b(t)$ depends only on $t>0$). Setting

$$g(t;\zeta) = \frac{b(t)}{(at)^2+\zeta^2},$$

we obtain the validity of condition I).

We now show that condition II) also holds:

$$\int\limits_{\mathbb{R}} |f(\xi)|g(t;x-\xi)d\xi \le \sum_{j=1}^{m} \int\limits_{|\xi-x_j|<1} |f(\xi)|g(t;x-\xi)d\xi + \int\limits_{\mathbb{R}\setminus\mathbb{U}} |f(\xi)|g(t;x-\xi)d\xi \le$$

$$\le \frac{cb(t)}{(at)^2}\sum_{j=1}^{m}\int\limits_{|\xi-x_j|<1}\frac{d\xi}{|\xi-x_j|^{\gamma_j}} + c_1 b(t)\int\limits_{\mathbb{R}}\frac{d\xi}{(at)^2+(x-\xi)^2} < +\infty, \quad x\in\mathbb{R}, \quad t>0.$$

The lemma is proved.

The main result is formulated as the following statement.

Theorem 1. *Let the initial function $f(\cdot)$ be continuous on $\mathbb{R}^n\setminus\{x_j\}_{j=1}^m$, bounded in a neighbourhood of the point at infinity, and satisfy estimates* (8) *in sufficiently small neighbourhoods of the points x_j. Then, on the set Π_+, formula* (11) *defines the unique solution of the Cauchy problem* (1), (9).

Proof. Since G_α is a solution of equation (1), using formulas (12) we obtain:

$$\partial_t u(t;x) = (f*\partial_t G_\alpha)(t;x) = -a(f*A_x^\alpha G_\alpha)(t;x) = -aA_x^\alpha u(t;x), \quad (t;x)\in\Pi_+.$$

Hence, on the set Π_+, representation (11) defines a classical solution of equation (1).

We now show that this solution satisfies the initial condition (9) in the sense of relation (13). Fix an arbitrary point $x\in\mathbb{R}^n\setminus\{x_j\}_{j=1}^m$, denote by l the index of the singular point x_j closest to x, and set $r := |x-x_l|$. According to formula (7), we have:

$$\Big|(f*G_\alpha)(t;x) - f(x)\Big| \le \int\limits_{\mathbb{R}^n} G_\alpha(t;\xi)|f(x-\xi)-f(x)|d\xi \equiv \mathfrak{L}(t;x).$$

Since f is continuous on $\mathbb{R}^n\setminus\{x_j\}_{j=1}^m$, then for every $\varepsilon>0$ there exists$t_0\in(0;r)$ such that $t_0^{\frac{1}{2(n+\alpha)}}<\varepsilon$ and $|f(x-\xi)-f(x)|<\varepsilon$, whenever $|\xi|<t_0^{\frac{1}{2(n+\alpha)}}$. Thus,

$$\mathfrak{L}(t;x) < \varepsilon \int\limits_{|\xi|<t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)d\xi + \int\limits_{|\xi|\ge t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)|f(x-\xi)-f(x)|d\xi \le$$

$$\leq \varepsilon \int_{\mathbb{R}^n} G_\alpha(t;\xi)d\xi + |f(x)| \int_{|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)d\xi + \int_{|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)|f(x-\xi)|d\xi \equiv$$
$$\equiv \varepsilon + \mathfrak{L}_0(t;x) + \mathfrak{L}_1(t;x),$$

where

$$\mathfrak{L}_0(t;x) := |f(x)| \int_{|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)d\xi, \quad \mathfrak{L}_1(t;x) := \int_{|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)|f(x-\xi)|d\xi.$$

Taking into account estimates (5) and (8), for $t>0$ we obtain:

$$\mathfrak{L}_0(t;x) \leq cr^{-\gamma_l}t \int_{|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} |\xi|^{-n-\alpha}d\xi \equiv \alpha\tilde{c}_1 t \int_{t_0^{\frac{1}{2(n+\alpha)}}}^{+\infty} \rho^{-1-\alpha}d\rho = \tilde{c}_1 t t_0^{-\frac{\alpha}{2(n+\alpha)}}. \tag{15}$$

To estimate the integral $\mathfrak{L}_1$ we use the decomposition

$$\mathfrak{L}_1(t;x) = \int_{2r\geq|\xi|\geq t_0^{\frac{1}{2(n+\alpha)}}} G_\alpha(t;\xi)|f(x-\xi)|d\xi + \int_{|\xi|\geq 2r} |G_\alpha(t;\xi)||f(x-\xi)|d\xi \equiv \mathfrak{L}_{11}(t;x) + \mathfrak{L}_{12}(t;x).$$

Once again, using (5) and (8), for $t>0$ we obtain:

$$\mathfrak{L}_{11}(t;x) \leq ct \int_{2r\geq|x-z|\geq t_0^{\frac{1}{2(n+\alpha)}}} \frac{|f(z)|dz}{\left(t^{1/\alpha}+|x-z|\right)^{n+\alpha}} \leq ctt_0^{-\frac{n+\alpha}{2(n+\alpha)}} \int_{2r\geq|x-z|} |f(z)|dz \equiv \tilde{c}_2 t t_0^{-\frac{1}{2}}; \tag{16}$$

$$\mathfrak{L}_{12}(t;x) \leq ct \int_{|\xi|\geq 2r} \frac{|f(x-\xi)|d\xi}{\left(t^{1/\alpha}+|\xi|\right)^{n+\alpha}} \leq ct \int_{|\xi|\geq 2r} \frac{|f(x-\xi)|d\xi}{|\xi|^{n+\alpha}} \equiv \tilde{c}_3 t. \tag{17}$$

From inequalities (15)–(17) it follows that for all $t \leq t_0$,

$$\mathfrak{L}_0(t;x) + \mathfrak{L}_1(t;x) \leq \tilde{c}_1 t_0^{\frac{2n+\alpha}{2(n+\alpha)}} + \tilde{c}_2 t_0^{\frac{1}{2}} + \tilde{c}_3 t_0 < \tilde{c}_1\varepsilon^{2n+\alpha} + \tilde{c}_2\varepsilon^{n+\alpha} + \tilde{c}_3\varepsilon^{2(n+\alpha)}.$$

Hence,

$$\forall x \in \mathbb{R}^n \setminus \{x_j\}_{j=1}^m \;\; \exists c>0 \;\; \forall \varepsilon>0 \;\; \exists t_0 < \varepsilon^{2(n+\alpha)} \;\; \forall t \leq t_0 :$$
$$\mathfrak{L}(t;x) < c(\varepsilon + \varepsilon^{n+\alpha} + \varepsilon^{2n+\alpha} + \varepsilon^{2(n+\alpha)}),$$

that is, the limit relation (13) holds.

Finally, according to the corresponding maximum principle for classical solutions of equation (1) from [2], the Cauchy problem (1), (9) cannot have more than one classical solution.

The theorem is proved.

3 Conclusions

It has been shown that the Cauchy problem for the isotropic superdiffusion equation (1) has a unique bounded smooth solution even in the case where the initial function has a finite number of discontinuities of the second kind. Thus, a generalization of known results for bounded continuous initial data is established.

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Літовченко В.А. *Розширення класу початкових даних задачі Коші для рівняння ізотропної супердифузії* // Буковинський матем. журнал — 2025. — Т.13, №2. — С. 177–187.

Розглядається задача Коші для рівняння ізотропної супердифузії з оператором Рісса дробового диференціювання порядку $\alpha \in (0;2)$, яке узагальнює класичне рівняння теплопровідності. Такі моделі застосовуються при описі перенесення енергії та маси у фрактальних і пористих середовищах, плазмових системах та інших складних структурах з аномально швидким поширенням. Оператор Рісса є генератором симетричних α-стійких процесів Леві, тому розв'язок задачі Коші може інтерпретуватися як густина ймовірності відповідного стохастичного процесу.

У класі обмежених гладких функцій встановлено існування класичного розв'язку задачі у випадку, коли початкова умова допускає скінченну кількість інтегровних розривів другого роду.