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ON INJECTIVE ENDOMORPHISMS OF THE SEMIGROUP $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ WITH THE TWO-ELEMENT FAMILY $\mathscr{F}^2$ OF INDUCTIVE NONEMPTY SUBSETS OF ω

We describe injective endomorphisms of the semigroup $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ with the two-element family $\mathscr{F}^2$ of inductive nonempty subsets of ω. In particular we show that every injective endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ is presented in the form $\mathfrak{e} = \mathfrak{e}_0\mathfrak{a}$, where $\mathfrak{e}_0$ is an injective $(0,0,[0))$-endomorphism of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ and $\mathfrak{a}$ is an automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Also we describe all injective $(0,0,[0))$-endomorphisms $\mathfrak{e}_0$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, i.e., such that $(0,0,[0))\mathfrak{e}_0 = (0,0,[0))$.

Key words and phrases: bicyclic monoid, extended bicyclic semigroup, inverse semigroup, bicyclic extension, endomorphism, injective.

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1 Introduction, motivation and main definitions

We shall follow the terminology of [1, 2, 13]. By ω we denote the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i+1 \in A$. Obvious, that $\varnothing$ is an inductive subset of ω.

Remark 1 ([6]). *1. By Lemma 6 from [5] a nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1+F)\cap F = F$.*

2. Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.

3. Statement (2) *implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.*

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n, m \in \omega$ we put $n-m+F = \{n-m+k \colon k \in F\}$ if $F \neq \varnothing$ and $n-m+\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega \colon x \geqslant a\}$.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an

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inverse semigroup, then the function inv: $S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [16].

The *bicyclic monoid* or the *bicyclic semigroup* $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_\omega = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1)\cdot(i_2,j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases} \tag{1}$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ to the semigroup $\boldsymbol{B}_\omega$ is isomorphic by the mapping $\mathfrak{h}\colon \mathscr{C}(p,q) \to \boldsymbol{B}_\omega$, $q^k p^l \mapsto (k,l)$ (see: [1, Section 1.12] or [15, Exercise IV.1.11(*ii*)]).

Next we shall describe the construction which is introduced in [5].

Let $\boldsymbol{B}_\omega$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases} \tag{2}$$

In [5] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i,j,\varnothing)\colon i,j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [5]. The semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [5] that $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particularly in [5] it is proved that the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [6]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [6], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described.

In [3, 14] the algebraic structure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω.

The set $\boldsymbol{B}_{\mathbb{Z}}=\mathbb{Z}\times\mathbb{Z}$ with the semigroup operation defined by formula (1) is called the *extended bicyclic semigroup* [17]. On the set $\boldsymbol{B}_{\mathbb{Z}}\times\mathscr{F}$, where $\mathscr{F}$ is an ω-closed subfamily of $\mathscr{P}(\omega)$, we define the semigroup operation "$\cdot$" by formula (2). In [7] it is proved that $(\boldsymbol{B}_{\mathbb{Z}}\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\mathbb{Z}\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\mathbb{Z}}\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}=\begin{cases}(\boldsymbol{B}_{\mathbb{Z}}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\mathbb{Z}}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{cases}$$

is defined in [7] similarly as in [5]. In [7] it is proven that $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is a combinatorial inverse semigroup. Green's relations, the natural partial order on the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, is isomorphic to the extended bicyclic semigroup, are derived. In particularly in [7] it is proved that the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_{Z}^{\mathscr{F}}$ is isomorphic to the extended bicyclic semigroup if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω. Also, in [7] it is proved that in the case when the family $\mathscr{F}$ consists of all singletons of ω and the empty set, the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the Brandt λ-extension of the semilattice $(\omega,\min)$, where $(\omega,\min)$ is the set ω with the semilattice operation $x\cdot y=\min\{x,y\}$.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_{\omega}$ is the identity self-map of $\boldsymbol{B}_{\omega}$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_{\omega})$ of automorphisms of $\boldsymbol{B}_{\omega}$ is trivial. The group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\boldsymbol{B}_{\mathbb{Z}}$ is established in [4] and there it is proved that $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}})$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. In the paper [9] we prove that for any family $\mathscr{F}$ of nonempty inductive subsets of ω the group $\mathbf{Aut}(\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}})$ of automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ is isomorphic to the additive group of integers.

In [12] the semigroups of endomorphisms of the bicyclic semigroup and the extended bicyclic semigroup are described. All types of monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ for two-element family $\mathscr{F}^2$ of nonempty inductive subsets of ω are described in [8, 10, 11].

This paper is a continuation of [7, 9]. We describe injective endomorphisms of the semigroup $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ with the two-element family $\mathscr{F}^2$ of inductive nonempty subsets of ω. In particular we show that every injective endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ is presented in the form $\mathfrak{e}=\mathfrak{e}_0\mathfrak{a}$, where $\mathfrak{e}_0$ is an injective $(0,0,[0))$-endomorphism of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ and $\mathfrak{a}$ is an automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Also we describe all injective $(0,0,[0))$-endomorphisms $\mathfrak{e}_0$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, i.e., such that $(0,0,[0))\mathfrak{e}_0=(0,0,[0))$.

Later we assume that an ω-closed family $\mathscr{F}^2$ consists of two inductive nonempty subsets of ω.

2 ENDOMORPHISMS OF THE SEMIGROUP $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ WITH THE FIXED POINT $(0,0,[0))$

If $\mathscr{F}$ is an arbitrary ω-closed family of inductive subsets in $\mathscr{P}(\omega)$ and $[s)\in\mathscr{F}$ for some $s\in\mathbb{Z}$ then

$$\boldsymbol{B}_{Z}^{\{[s)\}}=\{(i,j,[s))\colon i,j\in\mathbb{Z}\}$$

is a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ and by Proposition 5 of [7] the semigroup $\boldsymbol{B}_{Z}^{\{[s)\}}$ is isomorphic to the extended bicyclic semigroup.

Remark 2. *By Proposition 1 of [9] for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^*$ and the semigroups $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ and $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}^2$ contains the set $[0)$.*

An endomorphism $\mathfrak{e}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ is called a $(0,0,[0))$-*endomorphism* if $(0,0,[0))\mathfrak{e} = (0,0,[0))$.

Remark 3. *Theorem 1 of [8] state that for every injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ only one of the following conditions holds:*

(1) there exist a positive integer k and $p \in \{0,\ldots,k-1\}$ such that $\mathfrak{e} = \alpha_{k,p}$, where the mapping $\alpha_{k,p}$ defined by the formula

$$(i,j,[0))\alpha_{k,p} = (ki, kj, [0)),$$
$$(i,j,[1))\alpha_{k,p} = (p+ki, p+kj, [1)),$$

$i,j \in \omega$;

(2) there exist a positive integer $k \geqslant 2$ and $p \in \{1,\ldots,k-1\}$ such that $\mathfrak{e} = \beta_{k,p}$, where the mapping $\beta_{k,p}$ defined by the formula

$$(i,j,[0))\beta_{k,p} = (ki, kj, [0)),$$
$$(i,j,[1))\beta_{k,p} = (p+ki, p+kj, [0)),$$

$i,j \in \omega$.

For any integer k we define

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(k,k,0) = (k,k,[0)) \cdot \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2} \cdot (k,k,[0)).$$

By Proposition 2 [9], $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(k,k,0)$ is a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ which is isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$.

Fix an arbitrary positive integer k and any $p \in \{0,\ldots,k-1\}$. For all $i,j \in \mathbb{Z}$ we denote the transformation $\alpha_{k,p}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ in the following way

$$(i,j,[0))\alpha_{k,p} = (ki, kj, [0)),$$
$$(i,j,[1))\alpha_{k,p} = (p+ki, p+kj, [1)),$$

Lemma 1. *For an arbitrary positive integer k and any $p \in \{0,\ldots,k-1\}$ the map $\alpha_{k,p}$ is an injective endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$.*

Proof. By by Proposition 5 of [7] the subsemigroups $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ and $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$ are isomorphic to the extended bicyclic semigroup. By Proposition of [12] we have that the restrictions of $\alpha_{k,p}$ onto the subsemigroups $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ and $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$ are endomorphisms of $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ and $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$, respectively. This implies that for all $i,j,s,t \in \mathbb{Z}$ the following equalities hold

$$((i,j,[0)) \cdot (s,t,[0)))\alpha_{k,p} = (i,j,[0))\alpha_{k,p} \cdot (s,t,[0))\alpha_{k,p},$$
$$((i,j,[1)) \cdot (s,t,[1)))\alpha_{k,p} = (i,j,[1))\alpha_{k,p} \cdot (s,t,[1))\alpha_{k,p}.$$

For any $i,j,p,q\in\mathbb{Z}$ we have that

$$((i,j,[0))\cdot(s,t,[1)))\alpha_{k,p}=\begin{cases}(i+s-j,t,(j-s+[0))\cap[1))\alpha_{k,p}, & \text{if } j<s;\\ (i,t,[0)\cap[1))\alpha_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[0)\cap(s-j+[1)))\alpha_{k,p}, & \text{if } j>s\end{cases} =$$

$$=\begin{cases}(i+s-j,t,[1))\alpha_{k,p}, & \text{if } j<s;\\ (i,t,[1))\alpha_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[0))\alpha_{k,p}, & \text{if } j>s\end{cases} =$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[1)), & \text{if } j<s;\\ (p+ki,p+kt,[1)), & \text{if } j=s;\\ (ki,k(j+t-s),[0)), & \text{if } j>s,\end{cases}$$

$$(i,j,[0))\alpha_{k,p}\cdot(s,t,[1))\alpha_{k,p}=(ki,kj,[0))\cdot(p+ks,p+kt,[1))=$$

$$=\begin{cases}(ki+p+ks-kj,p+kt,(kj-p-ks+[0))\cap[1)), & \text{if } kj<p+ks;\\ (ki,p+kt,[0)\cap[1)), & \text{if } kj=p+ks;\\ (ki,kj+p+kt-p-ks,[0)\cap(p+ks-kj+[1))), & \text{if } kj>p+ks\end{cases} =$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[1)), & \text{if } kj<p+ks;\\ (ki,p+kt,[1)), & \text{if } kj=p+ks;\\ (ki,k(j+t-s),[0)), & \text{if } kj>p+ks\end{cases} =$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[1)), & \text{if } kj<ks;\\ (p+ki,p+kt,[1)), & \text{if } kj=ks;\\ (ki,kt,[1)), & \text{if } kj=p+ks \text{ and } p=0;\\ \texttt{vagueness}, & \text{if } kj=p+ks \text{ and } p\neq 0;\\ (ki,k(j+t-s),[0)), & \text{if } kj>ks\end{cases} =$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[1)), & \text{if } j<s;\\ (p+ki,p+kt,[1)), & \text{if } j=s;\\ (ki,k(j+t-s),[0)), & \text{if } j>s,\end{cases}$$

and

$$((i,j,[1))\cdot(s,t,[0)))\alpha_{k,p}=\begin{cases}(i+s-j,t,(j-s+[1))\cap[0))\alpha_{k,p}, & \text{if } j<s;\\ (i,t,[1)\cap[0))\alpha_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[1)\cap(s-j+[0)))\alpha_{k,p}, & \text{if } j>s\end{cases} =$$

$$=\begin{cases}(i+s-j,t,[0))\alpha_{k,p}, & \text{if } j<s;\\ (i,t,[1))\alpha_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[1))\alpha_{k,p}, & \text{if } j>s\end{cases} =$$

$$=\begin{cases}(k(i+s-j),kt,[0)), & \text{if } j<s;\\ (p+ki,p+kt,[1)), & \text{if } j=s;\\ (p+ki,p+k(j+t-s),[1)), & \text{if } j>s,\end{cases}$$

$$(i,j,[1))\alpha_{k,p}\cdot(s,t,[0))\alpha_{k,p}=(p+ki,p+kj,[1))\cdot(ks,kt,[0))=$$
$$=\begin{cases}(p+ki+ks-p-kj,kt,(p+kj-ks+[1))\cap[0)), & \text{if } p+kj<ks;\\ (p+ki,kt,[1)\cap[0)), & \text{if } p+kj=ks;\\ (p+ki,p+kj+kt-ks,[1)\cap(ks-p-kj+[0))), & \text{if } p+kj>ks\end{cases}=$$
$$=\begin{cases}(k(i+s-j),kt,[0)), & \text{if } p+kj<ks;\\ (p+ki,kt,[1)), & \text{if } p+kj=ks;\\ (p+ki,p+k(j+t-s),[1)), & \text{if } p+kj>ks\end{cases}=$$
$$=\begin{cases}(k(i+s-j),kt,[0)), & \text{if } kj<ks;\\ (ki,kt,[1)), & \text{if } p+kj=ks \text{ and } p=0;\\ \texttt{vagueness}, & \text{if } p+kj=ks \text{ and } p\neq 0;\\ (p+ki,p+kt,[1)), & \text{if } kj=ks;\\ (p+ki,p+k(j+t-s),[1)), & \text{if } kj>ks\end{cases}=$$
$$=\begin{cases}(k(i+s-j),kt,[0)), & \text{if } j<s;\\ (p+ki,p+kt,[1)), & \text{if } j=s;\\ (p+ki,p+k(j+t-s),[1)), & \text{if } j>s,\end{cases}$$

because $p\in\{0,\ldots,k-1\}$. Thus, $\alpha_{k,p}$ is an endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. □

Fix an arbitrary positive integer $k\geqslant 2$ and any $p\in\{1,\ldots,k-1\}$. For all $i,j\in\mathbb{Z}$ we define the transformation $\beta_{k,p}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ in the following way

$$(i,j,[0))\beta_{k,p}=(ki,kj,[0)),$$
$$(i,j,[1))\beta_{k,p}=(p+ki,p+kj,[0)).$$

It is obvious that $\beta_{k,p}$ is an injective transformation of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$.

Lemma 2. *For an arbitrary positive integer $k\geqslant 2$ and any $p\in\{1,\ldots,k-1\}$ the map $\beta_{k,p}$ is an injective endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$.*

Proof. By by Proposition 5 of [7] the subsemigroups $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ and $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$ are isomorphic to the extended bicyclic semigroup. By Lemma 3 of [12], the restriction of $\beta_{k,p}$ onto the subsemigroup $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ is an endomorphisms of $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$, and the restriction of $\beta_{k,p}$ onto the subsemigroup $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$ is a homomorphisms of $\boldsymbol{B}_{\mathbb{Z}}^{\{[1)\}}$ into $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$. This implies that for all $i,j,s,t\in\mathbb{Z}$ the following equalities hold

$$((i,j,[0))\cdot(s,t,[0)))\beta_{k,p}=(i,j,[0))\beta_{k,p}\cdot(s,t,[0))\beta_{k,p},$$
$$((i,j,[1))\cdot(s,t,[1)))\beta_{k,p}=(i,j,[1))\beta_{k,p}\cdot(s,t,[1))\beta_{k,p}.$$

For any $i,j,p,q\in\mathbb{Z}$ we have that

$$((i,j,[0))\cdot(s,t,[1)))\beta_{k,p}=\begin{cases}(i+s-j,t,(j-s+[0))\cap[1))\beta_{k,p}, & \text{if } j<s;\\ (i,t,[0)\cap[1))\beta_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[0)\cap(s-j+[1)))\beta_{k,p}, & \text{if } j>s\end{cases}=$$

$$=\begin{cases}(i+s-j,t,[1))\beta_{k,p}, & \text{if } j<s;\\ (i,t,[1))\beta_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[0))\beta_{k,p}, & \text{if } j>s\end{cases}=$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[0)), & \text{if } j<s;\\ (p+ki,p+kt,[0)), & \text{if } j=s;\\ (ki,k(j+t-s),[0)), & \text{if } j>s,\end{cases}$$

$$(i,j,[0))\beta_{k,p}\cdot(s,t,[1))\beta_{k,p}=(ki,kj,[0))\cdot(p+ks,p+kt,[0))=$$

$$=\begin{cases}(ki+p+ks-kj,p+kt,(kj-p-ks+[0))\cap[0)), & \text{if } kj<p+ks;\\ (ki,p+kt,[0)\cap[0)), & \text{if } kj=p+ks;\\ (ki,kj+p+kt-p-ks,[0)\cap(p+ks-kj+[0))), & \text{if } kj>p+ks\end{cases}=$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[0)), & \text{if } kj<p+ks;\\ (ki,p+kt,[0)), & \text{if } kj=p+ks;\\ (ki,k(j+t-s),[0)), & \text{if } kj>p+ks\end{cases}=$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[0)), & \text{if } kj<ks;\\ (p+ki,p+kt,[0)), & \text{if } kj=ks;\\ (ki,kt,[0)), & \text{if } kj=p+ks \text{ and } p=0;\\ \texttt{vagueness}, & \text{if } kj=p+ks \text{ and } p\neq 0;\\ (ki,k(j+t-s),[0))), & \text{if } kj>ks\end{cases}=$$

$$=\begin{cases}(p+k(i+s-j),p+kt,[0)), & \text{if } j<s;\\ (p+ki,p+kt,[0)), & \text{if } j=s;\\ (ki,k(j+t-s),[0)), & \text{if } j>s,\end{cases}$$

and

$$((i,j,[1))\cdot(s,t,[0)))\beta_{k,p}=\begin{cases}(i+s-j,t,(j-s+[1)\cap[0))\beta_{k,p}, & \text{if } j<s;\\ (i,t,[1)\cap[0))\beta_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[1)\cap(s-j+[0)))\beta_{k,p}, & \text{if } j>s\end{cases}=$$

$$=\begin{cases}(i+s-j,t,[0))\beta_{k,p}, & \text{if } j<s;\\ (i,t,[1))\beta_{k,p}, & \text{if } j=s;\\ (i,j+t-s,[1))\beta_{k,p}, & \text{if } j>s\end{cases}=$$

$$=\begin{cases}(k(i+s-j),kt,[0)), & \text{if } j<s;\\ (p+ki,p+kt,[0)), & \text{if } j=s;\\ (p+ki,p+k(j+t-s),[0)), & \text{if } j>s,\end{cases}$$

$$\begin{aligned}
&(i,j,[1])\beta_{k,p}\cdot(s,t,[0])\beta_{k,p}=(p+ki,p+kj,[0])\cdot(ks,kt,[0])=\\
&\quad=\begin{cases}(p+ki+ks-p-kj,kt,(p+kj-ks+[0])\cap[0]), & \text{if } p+kj<ks;\\ (p+ki,kt,[0]\cap[0]), & \text{if } p+kj=ks;\\ (p+ki,p+kj+kt-ks,[0]\cap(ks-p-kj+[0])), & \text{if } p+kj>ks\end{cases} =\\
&\quad=\begin{cases}(k(i+s-j),kt,[0]), & \text{if } p+kj<ks;\\ (p+ki,kt,[0]), & \text{if } p+kj=ks;\\ (p+ki,p+k(j+t-s),[0]), & \text{if } p+kj>ks\end{cases} =\\
&\quad=\begin{cases}(k(i+s-j),kt,[0]), & \text{if } kj<ks;\\ (ki,kt,[0]), & \text{if } p+kj=ks \text{ and } p=0;\\ \texttt{vagueness}, & \text{if } p+kj=ks \text{ and } p\neq 0;\\ (p+ki,p+kt,[0]), & \text{if } kj=ks;\\ (p+ki,p+k(j+t-s),[0]), & \text{if } kj>ks\end{cases} =\\
&\quad=\begin{cases}(k(i+s-j),kt,[0]), & \text{if } j<s;\\ (p+ki,p+kt,[0]), & \text{if } j=s;\\ (p+ki,p+k(j+t-s),[0]), & \text{if } j>s,\end{cases}
\end{aligned}$$

because $p\in\{1,\ldots,k-1\}$. Thus, $\beta_{k,p}$ is an endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. □

Lemma 3. *Let $\mathfrak{e}$ be a $(0,0,[0])$-endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Then there exists a non-negative integer n such that $(i,j,[0])\mathfrak{e}=(ni,nj,[0])$ for all $i,j\in\mathbb{Z}$.*

Proof. By by Proposition 5 of [7] the subsemigroup $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ is isomorphic to the extended bicyclic semigroup. By Lemma 3 of [12], the restriction of the transformation $\mathfrak{e}$ onto the subsemigroup $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$ is an endomorphisms of $\boldsymbol{B}_{\mathbb{Z}}^{\{[0)\}}$. Then Lemma 5 of [12] implies the statement of the lemma. □

Theorem 1. *Let $\mathfrak{e}$ be an injective $(0,0,[0])$-endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Then one of the following conditions holds:*

(1) there exist a positive integer k and $p\in\{0,\ldots,k-1\}$ such that $\mathfrak{e}=\alpha_{k,p}$;

(2) there exist a positive integer $k\geqslant 2$ and $p\in\{1,\ldots,k-1\}$ such that $\mathfrak{e}=\beta_{k,p}$.

Proof. It is obvious that for any $(i,j,[l))\in\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, $l=0,1$, there exists a non-negative integer n such that $(i,j,[l))\in\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)$. This implies the equality

$$\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}=\bigcup_{n\in\omega}\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0).$$

Also by the semigroup operation of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ for $m,n\in\omega$ we have that $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)\subseteq\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-m,-m,0)$ if and only if $m\geqslant n$.

Since $\mathfrak{e}$ is an injective $(0,0,[0])$-endomorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, Lemma 3 implies that there exists a positive integer k such that $(i,j,[0))\mathfrak{e}=(ki,kj,[0))$ for all $i,j\in\mathbb{Z}$.

Fix an arbitrary positive integer n. By Proposition 2 [9], $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)$ is a subsemigroup of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ which is isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$. This implies that the semigroups $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)$ and $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0)$ are isomorphic. By Corollary 2 from [6] every automorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ is the identity map, and hence every automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0)$ is the identity map, too.

We define the isomorphism $\mathfrak{I}_0^{-n}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0) \to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0)$ by the formula

$$(i-n, j-n, [s])\mathfrak{I}_0^{-n} = (i, j, [s]),$$

for any positive integers i, j and $s \in \{0,1\}$. The above arguments imply that so defined isomorphism is unique. Hence we have that for any injective endomorphism $\mathfrak{e}_{-n}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)$ there exists an injective endomorphism $\mathfrak{e}_0$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0)$ such that the following diagram

$$\begin{array}{ccc}
\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0) & \xrightarrow{\ \mathfrak{e}_{-n}\ } & \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0) \\
\Big\downarrow{\scriptstyle \mathfrak{I}_0^{-n}} & & \Big\downarrow{\scriptstyle \mathfrak{I}_0^{-n}} \\
\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0) & \overset{}{\underset{\mathfrak{e}_0}{\dashrightarrow}} & \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(0,0,0)
\end{array}$$

is commutative. Hence, by Remark 3 we have that for any injective endomorphism $\mathfrak{e}_{-n}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}(-n,-n,0)$ one of the following conditions holds:

(1) there exist a positive integer k and $p \in \{0, \ldots, k-1\}$ such that $\mathfrak{e}_0 = \alpha_{k,p}$;

(2) there exist a positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$ such that $\mathfrak{e}_0 = \beta_{k,p}$.

If $\mathfrak{e}_0 = \alpha_{k,p}$ then

$$\begin{aligned}
(i-n, j-n, [0])\mathfrak{e}_{-n} &= (((i-n, j-n, [0])\mathfrak{I}_0^{-n})\alpha_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= ((i, j, [0])\alpha_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= (ki, kj, [0])(\mathfrak{I}_0^{-n})^{-1} = \\
&= (ki-n, kj-n, [0])
\end{aligned}$$

and

$$\begin{aligned}
(i-n, j-n, [1])\mathfrak{e}_{-n} &= (((i-n, j-n, [1])\mathfrak{I}_0^{-n})\alpha_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= ((i, j, [0])\alpha_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= (p+ki, p+kj, [1])(\mathfrak{I}_0^{-n})^{-1} = \\
&= (p+ki-n, p+kj-n, [1]),
\end{aligned}$$

for any positive integers i, j.

If $\mathfrak{e}_0 = \beta_{k,p}$ then

$$\begin{aligned}
(i-n, j-n, [0])\mathfrak{e}_{-n} &= (((i-n, j-n, [0])\mathfrak{I}_0^{-n})\beta_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= ((i, j, [0])\beta_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= (ki, kj, [0])(\mathfrak{I}_0^{-n})^{-1} = \\
&= (ki-n, kj-n, [0])
\end{aligned}$$

and

$$\begin{aligned}
(i-n, j-n, [1])\mathfrak{e}_{-n} &= (((i-n, j-n, [1])\mathfrak{I}_0^{-n})\beta_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= ((i, j, [0])\beta_{k,p})(\mathfrak{I}_0^{-n})^{-1} = \\
&= (p+ki, p+kj, [0])(\mathfrak{I}_0^{-n})^{-1} = \\
&= (p+ki-n, p+kj-n, [0]),
\end{aligned}$$

for any positive integers i,j.

This completes the proof of the theorem. □

3 On injective endomorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$

Remark 4. *1. By Theorem 1 of [9] every $(0,0,[0])$-automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ is the identity map.*

2. For every integer s the map $\mathfrak{h}_s\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}\to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, $(i,j,[p])\mapsto(i+s,j+s,[q])$, $i,j\in\mathbb{Z}$, $q\in\{0,1\}$, is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ (Proposition 6 of [9]).

3. The map $\widetilde{\mathfrak{a}}\colon \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}\to \boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, $(i,j,[p])\mapsto(i+q,j+q,[1-q])$, $i,j\in\mathbb{Z}$, $q\in\{0,1\}$, is an automorphism of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ (Lemma 2 of [9]), and moreover $\widetilde{\mathfrak{a}}^2=\mathfrak{h}_1$.

Lemma 4. *For any endomorphism $\mathfrak{e}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ there exists an automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ such that $(0,0,[0])\mathfrak{e}=(0,0,[0])\mathfrak{a}$. Moreover, $\mathfrak{a}=\widetilde{\mathfrak{a}}^{2s}=\mathfrak{h}_s$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[0])$, and $\mathfrak{a}=\widetilde{\mathfrak{a}}^{2s+1}=\mathfrak{h}_s\widetilde{\mathfrak{a}}$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[1])$ for some integer s.*

Proof. Since any homomorphic image of an idempotent is again an idempotent, by Lemma 1 of [7] there exist an integer s and $q\in\{0,1\}$ such that $(0,0,[0])\mathfrak{e}=(s,s,[q])$. Simple verifications and Remark 4 imply that $(0,0,[0])\widetilde{\mathfrak{a}}^{2s}=0,0,[0])=(s,s,[0])$ and $(0,0,[0])\widetilde{\mathfrak{a}}^{2s+1}=(0,0,[0])(\mathfrak{h}_s\widetilde{\mathfrak{a}})=(s,s,[1])$. □

Theorem 2. *For any endomorphism $\mathfrak{e}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ there exist a $(0,0,[0])$-endomorphism $\mathfrak{e}_0$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ and an automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ such that $\mathfrak{e}=\mathfrak{e}_0\mathfrak{a}$. Moreover, $\mathfrak{e}=\mathfrak{e}_0\widetilde{\mathfrak{a}}^{2s}=\mathfrak{e}_0\mathfrak{h}_s$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[0])$, and $\mathfrak{e}=\mathfrak{e}_0\widetilde{\mathfrak{a}}^{2s+1}=\mathfrak{e}_0\mathfrak{h}_s\widetilde{\mathfrak{a}}$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[1])$ for some integer s.*

Proof. By Lemma 4 there exists an automorphism $\mathfrak{a}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ such that $(0,0,[0])\mathfrak{e}=(0,0,[0])\mathfrak{a}$. Then the product $\mathfrak{e}\mathfrak{a}^{-1}$ is a $(0,0,[0])$-endomorphism of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Let be $\mathfrak{e}_0=\mathfrak{e}\mathfrak{a}^{-1}$. Since for an arbitrary monoid S every right translation $\rho_u\colon S\to S$, $s\mapsto su$ on an element of the group of units of S is a bijective map, we conclude that the equality $\mathfrak{e}_0=\mathfrak{e}\mathfrak{a}^{-1}$ implies that $\mathfrak{e}=\mathfrak{e}_0\mathfrak{a}$. The last statement follows from the second statemnet of Lemma 4. □

Since the composition of two injective maps is an injective map, Theorems 1 and 2 imply the following theorem, which describes the structure of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$.

Theorem 3. *For any injective endomorphism $\mathfrak{e}$ of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ there exist an injective $(0,0,[0])$-endomorphism $\mathfrak{e}_0$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ and an automorphism $\mathfrak{a}$ of $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ such that $\mathfrak{e}=\mathfrak{e}_0\mathfrak{a}$. Moreover, $\mathfrak{e}=\mathfrak{e}_0\widetilde{\mathfrak{a}}^{2s}=\mathfrak{e}_0\mathfrak{h}_s$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[0])$, $\mathfrak{e}=\mathfrak{e}_0\widetilde{\mathfrak{a}}^{2s+1}=\mathfrak{e}_0\mathfrak{h}_s\widetilde{\mathfrak{a}}$ in the case when $(0,0,[0])\mathfrak{e}=(s,s,[1])$ for some integer s, and one of the following conditions holds:*

(1) there exist a positive integer k and $p\in\{0,\ldots,k-1\}$ such that $\mathfrak{e}_0=\alpha_{k,p}$;

(2) there exist a positive integer $k\geqslant 2$ and $p\in\{1,\ldots,k-1\}$ such that $\mathfrak{e}_0=\beta_{k,p}$.

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Гутік О., Позднякова І. *Про ін'єктивні ендоморфізми напівгрупи* $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ *з двоелементною сім'єю* $\mathscr{F}^2$ *індуктивних непорожніх підмножин у* ω // Буковинський матем. журнал — 2025. — Т.13, №2. — С. 58–69.

Ми описуємо ін'єктивні ендоморфізми напівгрупи $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$ з двоелементною сім'єю $\mathscr{F}^2$ індуктивних непорожніх підмножин у ω. Зокрема, доводимо, що кожний ін'єктивний ендоморфізм $\mathfrak{e}$ напівгрупи $\boldsymbol{B}_{Z}^{\mathscr{F}^2}$ зображається у вигляді $\mathfrak{e} = \mathfrak{e}_0\mathfrak{a}$, де $\mathfrak{e}_0$ – ін'єктивний $(0, 0, [0])$-ендоморфізм і $\mathfrak{a}$ – автоморфізм $\mathfrak{a}$ напівгрупи $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$. Також ми описуємо усі ін'єктивні $(0, 0, [0])$-ендоморфізми $\mathfrak{e}_0$ напівгрупи $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}^2}$, тобто такі, що $(0, 0, [0])\mathfrak{e}_0 = (0, 0, [0])$.