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Moklyachuk M.P., Masyutka O.Yu.

INTERPOLATION PROBLEM FOR MULTIDIMENSIONAL STATIONARY RANDOM FIELD

In the article we propose methods of the mean-square optimal linear interpolation of the functional which depend on the unknown values of the multidimensional stationary random field based on observed data of the field with noise. Under condition of spectral certainty when the spectral densities of the stationary fields are known we derive formulas for calculating the spectral characteristics and the mean-square errors of the estimates of the functional. Analogous results are derived for the case of observations of the field without noise. In the case of spectral uncertainty when certain sets of admissible densities are given we derive the relations that the least spectral densities satisfy.

Key words and phrases: multidimensional stationary random field, optimal linear estimate, minimax-robust estimate, minimax spectral characteristic, least favorable spectral density.

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine (Moklyachuk M.P.)
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine (Masyutka O.Yu.)
e-mail: *moklyachuk@gmail.com (Moklyachuk M.P.), omasyutka@gmail.com (Masyutka O.Yu.)*

Introduction

Investigation of the properties of stationary stochastic processes plays an important role both in the theory of stochastic processes and application it to the practice. A great number of scientific papers deals with the problem of estimation of unknown values of a stationary process. The formulation and effective methods of solution of the problems of interpolation, extrapolation and filtering of stationary sequences and processes belong to A. N. Kolmogorov [9]. Further analysis can be found in E. J. Hannan [4]. A significant contribution to the theory of forecasting was made by H. Wold [19, 20], T. Nakazi [14]. Constructive methods of solution of estimation problems for stationary stochastic sequences and processes were developed by N. Wiener [18]. For more results and references see also [6], [8], [10], [21].

The basic assumption of most of the methods of estimation of the unobserved values of stochastic processes is that the spectral densities of the considered stochastic processes are exactly known. However, in practice, these methods are not applicable since the complete information on the spectral densities is impossible in most cases. In order to solve the problem parametric or nonparametric estimates of the unknown spectral densities are found. Then, one of traditional estimation methods

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is applied, provided that the selected densities are the true ones. This procedure can result in significant increasing of the value of error as K. S. Vastola and H. V. Poor [17] have demonstrated with the help of some examples. To avoid this effect one can search the estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error. The paper by Ulf Grenander [3] should be marked as the first one where this approach to extrapolation problem for stationary processes was proposed.

Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H. V. Poor [7]. J. Franke [1], J. Franke and H. V. Poor [2] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities.

In the book by M. Moklyachuk and O. Masyutka a minimax technique of the estimation for vector-valued stationary stochastic processes is proposed [11]. Estimation problems for random fields was considered in the paper by M. Moklyachuk and N. Shchestyuk [12]. In the book by M. Moklyachuk, O. Masyutka, I. Golichenko [13] the minimax approach is applied to investigate the estimation problems for functionals which depend on the unknown values of stationary random fields on a sphere.

In this article we present results of investigation of the problem of the mean-square optimal estimation of the linear functional

$$A_K\vec{\xi} = \sum_{(k,j)\in K} (\vec{a}(k,j))^\top \vec{\xi}(k,j)$$

which depend on the unknown values of a multidimensional stationary random field $\vec{\xi}(k,j) = \{\xi_i(k,j)\}_{i=1}^T$, $k,j \in Z$ from observations of the field $\vec{\xi}(k,j)+\vec{\eta}(k,j)$ at points $(k,j) \in Z^2 \setminus K$ where $\vec{\eta}(k,j) = \{\eta_i(k,j)\}_{i=1}^T$ is an uncorrelated with $\vec{\xi}(k,j)$ multidimensional stationary random field. The problem is investigated in the case of spectral certainty, where the spectral densities of the fields are exactly known, and in the case of spectral uncertainty, where the spectral densities of the fields are unknown while a class of admissible spectral densities is given.

1 Classical interpolation problem for stationary fields

Let $\vec{\xi}(k,j) = \{\xi_i(k,j)\}_{i=1}^T$, $k,j \in Z$ and $\vec{\eta}(k,j) = \{\eta_i(k,j)\}_{i=1}^T$, $k,j \in Z$ be uncorrelated multidimensional stationary random fields. We will consider $\xi_i(k,j)$, $\eta_i(k,j)$ as elements of the Hilbert space $H = L_2(\Omega, P)$ of complex valued random variables with zero first moment, $E\xi = 0$, finite second moment, $E|\xi|^2 < \infty$, and the inner product $(\xi,\eta) = E\xi\overline{\eta}$. We will consider stationary stochastic fields with absolutely continuous spectral measures and the correlation functions

$$R_\xi(m,n) = E\vec{\xi}(k+m,j+n)(\vec{\xi}(k,j))^*,\ R_\eta(m,n) = E\vec{\eta}(k+m,j+n)(\vec{\eta}(k,j))^*$$

of the stationary stochastic fields $\vec{\xi}(k,j)$, $\vec{\eta}(k,j)$, $k,j \in \mathbb{Z}$, that admit the spectral decompositions [5]

$$R_\xi(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(m\lambda+n\mu)}F(\lambda,\mu)d\lambda d\mu,\ R_\eta(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(m\lambda+n\mu)}G(\lambda,\mu)d\lambda d\mu,$$

where $F(\lambda,\mu)=\{f_{ij}(\lambda,\mu)\}_{i,j=1}^{T}$ and $G(\lambda,\mu)=\{g_{ij}(\lambda,\mu)\}_{i,j=1}^{T}$ are the spectral density functions of the fields $\vec{\xi}(k,j)$, $\vec{\eta}(k,j)$, $k,j\in\mathbb{Z}$ that satisfiy the minimality condition

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\left[(F(\lambda,\mu)+G(\lambda,\mu))^{-1}\right]d\lambda d\mu\;<\;\infty. \qquad (1)$$

This condition is necessary and sufficient in order that the error-free interpolation of unknown values of the field is impossible [4].

Consider the problem of the mean-square optimal linear estimation of the functional

$$A_K\vec{\xi}=\sum_{(k,j)\in K}(\vec{a}(k,j))^{\top}\vec{\xi}(k,j)$$

which depend on the unknown values of a multidimensional stationary random field $\vec{\xi}(k,j)$ from observations of the field $\vec{\xi}(k,j)+\vec{\eta}(k,j)$ at points $(k,j)\in Z^2\backslash K$. We will assume that the coefficients $\vec{a}(k,j),(k,j)\in K$ satisfy the following condition

$$\sum_{(k,j)\in K}\|\vec{a}(k,j)\|<\infty.$$

Under this condition the functional $A_K\vec{\xi}$ has finite second moment.

We can represent the fuctional $A_K\vec{\xi}$ in the form

$$A_K\vec{\xi}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(A_K(\lambda,\mu))^{\top}(Z_{\xi}(d\lambda,d\mu)+Z_{\eta}(d\lambda,d\mu)),$$

where $Z_{\xi}(d\lambda,d\mu)$ and $Z_{\eta}(d\lambda,d\mu)$ are vector-valued orthogonal stochastic measures of the fields $\vec{\xi}(k,j)$ and $\vec{\eta}(k,j)$ and

$$A_K(\lambda,\mu)=\sum_{(k,j)\in K}\vec{a}(k,j)e^{i(k\lambda+j\mu)}.$$

Denote by $\hat{A}_K\vec{\xi}$ the optimal linear estimate of the functional $A_K\vec{\xi}$ from observations of the field $\vec{\xi}(k,j)+\vec{\eta}(k,j)$ at points $(k,j)\in Z^2\setminus K$ and by $\Delta(F,G)=E\left|A_K\vec{\xi}-\hat{A}_K\vec{\xi}\right|^2$ the mean-square error of the estimate $\hat{A}_K\vec{\xi}$. Since spectral densities of stationary fields $\vec{\xi}(k,j)$ and $\vec{\eta}(k,j)$ are known, we can use the method of orthogonal projections in Hilbert spaces [9] to find the estimate $\hat{A}_K\vec{\xi}$.

Denote by $H_K^{-}(\xi+\eta)$ the closed linear subspace generated by elements

$$\{\xi_i(k,j)+\eta_i(k,j):(k,j)\in Z^2\setminus K, i=1,....,T\}$$

in the Hilbert space $H=L_2(\Omega,P)$. Denote by $L_2(F+G)$ the Hilbert space of functions $\vec{a}(\lambda,\mu)$ such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\vec{a}(\lambda,\mu))^{\top}(F(\lambda,\mu)+G(\lambda,\mu))\overline{\vec{a}(\lambda,\mu)}d\lambda d\mu<\infty,$$

and denote by $L_2^{K-}(F+G)$ the subspace of the space $L_2(F+G)$ generated by the functions

$$\{e^{i(k\lambda+j\mu)}\delta_i,\delta_i=\{\delta_{il}\}_{l=1}^{T},(k,j)\in Z^2\setminus K,i=1,....,T\}.$$

The mean-square optimal linear estimate $\hat{A}_K\vec{\xi}$ of the functional $A_K\vec{\xi}$ can be represented in the form

$$\hat{A}_K\vec{\xi}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(h(\lambda,\mu))^{\top}(Z_{\xi}(d\lambda,d\mu)+Z_{\eta}(d\lambda,d\mu)),$$

where $h(\lambda,\mu)\in L_2^{K-}(F+G)$ is the spectral characteristic of the estimate.

The mean-square error $\Delta(h;F,G)$ of the estimate $\hat{A}_K\vec{\xi}$ is given by the formula

$$\Delta(h;F,G)=E\left|A_K\vec{\xi}-\hat{A}_K\vec{\xi}\right|^2$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(A_K(\lambda,\mu)-h(\lambda,\mu))^\top F(\lambda,\mu)(\overline{A_K(\lambda,\mu)-h(\lambda,\mu)})d\lambda d\mu$$

$$+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(h(\lambda,\mu))^\top G(\lambda,\mu)\overline{h(\lambda,\mu)}d\lambda d\mu.$$

According to the Hilbert space projection method proposed by A. N. Kolmogorov [9], the optimal estimation of the functional $A_K\vec{\xi}$ is a projection of the element $A_K\vec{\xi}$ of the space H on the space $H_K^-(\xi+\eta)$. It can be found from the following conditions:

$$1)\hat{A}_K\vec{\xi}\in H_K^-(\xi+\eta),$$
$$2)A_K\vec{\xi}-\hat{A}_K\vec{\xi}\perp H_K^-(\xi+\eta).$$

It follows from the second condition that the spectral characteristic $h(\lambda,\mu)$ of the optimal linear estimate $\hat{A}_K\vec{\xi}$ for any $(k,j)\in Z^2\setminus K$ satisfies equations

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(A_K(\lambda,\mu)-h(\lambda,\mu)\right)^\top F(\lambda,\mu)e^{-i(k\lambda+j\mu)}d\lambda d\mu$$

$$-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(h(\lambda,\mu))^\top G(\lambda,\mu)e^{-i(k\lambda+j\mu)}d\lambda d\mu=0.$$

The last relation is equivalent to equations

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((A_K(\lambda,\mu))^\top F(\lambda,\mu)-(h(\lambda,\mu))^\top(F(\lambda,\mu)+G(\lambda,\mu))e^{-i(k\lambda+j\mu)}d\lambda d\mu=0, \tag{2}$$

where $(k,j)\in Z^2\setminus K$. Define the function

$$(C_K(\lambda,\mu))^\top=(A_K(\lambda,\mu))^\top F(\lambda,\mu)-(h(\lambda,\mu))^\top(F(\lambda,\mu)+G(\lambda,\mu))$$

and its Fourier coefficients

$$\vec{c}(k,j)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_K(\lambda,\mu)e^{-i(k\lambda+j\mu)}d\lambda d\mu,(k,j)\in Z^2.$$

It follows from the condition (2) that the coefficients $\vec{c}(k,j)$ is nonzero only on the set K. Hence,

$$C_K(\lambda,\mu)=\sum_{(k,j)\in K}\vec{c}(k,j)e^{i(k\lambda+j\mu)}$$

and the spectral characteristic of the estimate $\hat{A}_K\vec{\xi}$ is of the form

$$(h(F,G))^\top=(A_K(\lambda,\mu))^\top F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}$$
$$-(C_K(\lambda,\mu))^\top(F(\lambda,\mu)+G(\lambda,\mu))^{-1}. \tag{3}$$

It follows from the first condition, $\hat{A}_K\vec{\xi}\in H_K^-(\xi+\eta)$, which determine the optimal linear estimate of the functional $A_K\vec{\xi}$, that the following relation holds true

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left((A_K(\lambda,\mu))^\top F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}\right)e^{-i(k\lambda+j\mu)}d\lambda d\mu$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big((C_K(\lambda,\mu))^\top (F(\lambda,\mu)+G(\lambda,\mu))^{-1}\Big)\, e^{-i(k\lambda+j\mu)}d\lambda d\mu, (k,j)\in K. \quad (4)$$

Let us determine the Fourier coefficients of the matrix-valued functions

$$((F(\lambda,\mu)+G(\lambda,\mu))^{-1})^\top, (F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1})^\top,$$

$$(F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}G(\lambda,\mu))^\top$$

as follows

$$B(k,j) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((F(\lambda,\mu)+G(\lambda,\mu))^{-1})^\top e^{-i(k\lambda+j\mu)}d\lambda d\mu,$$

$$D(k,j) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1})^\top e^{-i(k\lambda+j\mu)}d\lambda d\mu,$$

$$R(k,j) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}G(\lambda,\mu))^\top e^{-i(k\lambda+j\mu)}d\lambda d\mu.$$

The relation (4) can be rewritten in the form

$$\sum_{(m,n)\in K} D(k-m,j-n)\vec{a}(m,n) = \sum_{(m,n)\in K} B(k-m,j-n)\vec{c}(m,n), (k,j)\in K. \quad (5)$$

The mean-square error of the estimate $\hat{A}_K\vec{\xi}$ can be calculated by the formula

$$\Delta(F,G) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((A_K(\lambda,\mu))^\top G(\lambda,\mu)+(C_K(\lambda,\mu))^\top)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}F(\lambda,\mu)$$
$$\times(F(\lambda,\mu)+G(\lambda,\mu))^{-1}((A_K(\lambda,\mu))^\top G(\lambda,\mu)+(C_K(\lambda,\mu))^\top)^* d\lambda d\mu$$
$$+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((A_K(\lambda,\mu))^\top F(\lambda,\mu)-(C_K(\lambda,\mu))^\top)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}G(\lambda,\mu)$$
$$\times (F(\lambda,\mu)+G(\lambda,\mu))^{-1}((A_K(\lambda,\mu))^\top F(\lambda,\mu)-(C_K(\lambda,\mu))^\top)^* d\lambda d\mu. \quad (6)$$

Let us summarize results and present them in the form of a theorem.

Theorem 1. *Let $\vec{\xi}(k,j)$ and $\vec{\eta}(k,j)$ be uncorrelated T-dimensional stationary fields with spectral density matrices $F(\lambda,\mu)$ and $G(\lambda,\mu)$ which satisfy the minimality condition (1). The spectral characteristics $h(F,G)$ and the mean-square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A_K\vec{\xi}$ which depends on the unknown values of the field $\vec{\xi}(k,j)$ based on observations of the field $\vec{\xi}(k,j)+\vec{\eta}(k,j)$, $k,j)\in Z^2\setminus K$ can be calculated by formulas (3), (6).*

Corollary 1. *Let $K=\{(k,j): k\in Z, 0\le j\le N\}$. The spectral characteristics $h_N(F,G)$ of the optimal linear estimate of the functional*

$$A_N\vec{\xi} = \sum_{k=-\infty}^{\infty}\sum_{j=0}^{N}(\vec{a}(k,j))^\top\vec{\xi}(k,j)$$

which depends on the unknown values of the field $\vec{\xi}(k,j)$ based on observations of the field $\vec{\xi}(k,j)+\vec{\eta}(k,j)$, $(k,j)\in Z^2\setminus K$ can be calculated by formula (3) where

$$A_K(\lambda,\mu) = A_N(\lambda,\mu) = \sum_{k=-\infty}^{\infty}\sum_{j=0}^{N}\vec{a}(k,j)e^{i(k\lambda+j\mu)} = \sum_{j=0}^{N}\vec{a}_j(\lambda)e^{ij\mu},\ \vec{a}_j(\lambda) = \sum_{k=-\infty}^{\infty}\vec{a}(k,j)e^{ik\lambda},$$

$$C_K(\lambda,\mu)=C_N(\lambda,\mu)=\sum_{k=-\infty}^{\infty}\sum_{j=0}^{N}\vec{c}(k,j)e^{i(k\lambda+j\mu)}=\sum_{j=0}^{N}\vec{c}_j(\lambda)e^{ij\mu},\ \vec{c}_j(\lambda)=\sum_{k=-\infty}^{\infty}\vec{c}(k,j)e^{ik\lambda},$$

and vector of unknown coefficients $\vec{c}_N(\lambda)=\{\vec{c}_j(\lambda)\}_{j=0}^{N}$ *is determined by relation*

$$\vec{c}_N(\lambda)=B_N^{-1}(\lambda)D_N(\lambda)\vec{a}_N(\lambda) \tag{7}$$

where $\vec{a}_N(\lambda)=\{\vec{a}_j(\lambda)\}_{j=0}^{N}$ *and* $B_N(\lambda)$, $D_N(\lambda)$ *are matrices constructed from block-matrices of dimension* $T\times T$

$$B_N(\lambda)(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}((F(\lambda,\mu)+G(\lambda,\mu))^{-1})^{\top}e^{i(j-k)\mu}d\mu,$$

$$D_N(\lambda)(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1})^{\top}e^{i(j-k)\mu}d\mu,\ k,j=0,1,....,N.$$

The mean-square error $\Delta_N(F,G)$ *of the optimal linear estimate of the functional* $A_N\vec{\xi}$ *can be calculated by formula*

$$\Delta_N(F,G)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(<\vec{a}_N(\lambda),R_N(\lambda)\vec{a}_N(\lambda)>+<\vec{c}_N(\lambda),B_N(\lambda)\vec{c}_N(\lambda)>)d\lambda, \tag{8}$$

where $R_N(\lambda)$ *is a matrix constructed from block-matrices of dimension* $T\times T$

$$R_N(\lambda)(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\lambda,\mu)(F(\lambda,\mu)+G(\lambda,\mu))^{-1}G(\lambda,\mu))^{\top}e^{i(j-k)\mu}d\mu,\ k,j=0,1,....,N,$$

and $<a,b>$ *is the inner product.*

Corollary 2. *Let* $\vec{\xi}(k,j)$ *be T-dimensional stationary field with spectral density matrix* $F(\lambda,\mu)$ *which satisfy the minimality condition*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\left[(F(\lambda,\mu))^{-1}\right]d\lambda d\mu<\infty. \tag{9}$$

The spectral characteristics $h_N(F)$ *and the mean-square error* $\Delta_N(F)$ *of the optimal linear estimate of the functional* $A_N\vec{\xi}$ *which depends on the unknown values of the field* $\vec{\xi}(k,j)$ *based on observations of the field*

$$\vec{\xi}(k,j),\ (k,j)\in Z^2\setminus K,\ K=\{(k,j):k\in Z,0\le j\le N\}$$

can be calculated by formulas

$$(h_N(F))^{\top}=(A_N(\lambda,\mu))^{\top}-(C_N(\lambda,\mu))^{\top}(F(\lambda,\mu))^{-1}, \tag{10}$$

$$\Delta_N(F)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(C_N(\lambda,\mu))^{\top}(F(\lambda,\mu))^{-1}\overline{C_N(\lambda,\mu)}d\lambda d\mu$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}<\vec{a}_N(\lambda),B_N^{-1}(\lambda)\vec{a}_N(\lambda)>d\lambda, \tag{11}$$

where

$$C_N(\lambda,\mu)=\sum_{j=0}^{N}\vec{c}_j(\lambda)e^{ij\mu},\ \vec{c}_N(\lambda)=B_N^{-1}(\lambda)\vec{a}_N(\lambda),$$

$B_N(\lambda)$ *is a matrix constructed from block-matrices*

$$B_N(\lambda)(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}((F(\lambda,\mu))^{-1})^{\top}e^{i(j-k)\mu}d\mu,\ k,j=0,1,....,N.$$

Example 1. *Let's investigate the problem of linear estimation of the functional*

$$\vec{a}(0,0)^{\top}\vec{\xi}(0,0)+\vec{a}(0,1)^{\top}\vec{\xi}(0,1)$$

based on observations of the field $\vec{\xi}(k,j)=\{\xi_i(k,j)\}_{i=1}^{2}$ *where* $\xi_1(k,j)=\xi(k,j)$ *is a stationary stochastic field with the spectral density* $f(\lambda,\mu)$*, and* $\xi_2(k,j)=\xi(k,j)+\eta(k,j)$*, where* $\eta(k,j)$ *is an uncorrelated with* $\xi(k,j)$ *stationary stochastic field with the spectral density* $g(\lambda,\mu)$*. Consider this problem for the stationary fields with the spectral densities*

$$f(\lambda,\mu)=\frac{1}{|1-b_1e^{i\lambda}|^2\,|1-b_2e^{i\mu}|^2},\quad g(\lambda,\mu)=\frac{1}{|1-b_1e^{i\lambda}|^2},\ b_1,b_2\in R,\ |b_1|<1,\ |b_2|<1.$$

In this case the matrix of spectral densities is of the form

$$F(\lambda,\mu)=\begin{pmatrix} f(\lambda,\mu) & f(\lambda,\mu)\\ f(\lambda,\mu) & f(\lambda,\mu)+g(\lambda,\mu)\end{pmatrix}.$$

Its determinant equals

$$D=|F(\lambda,\mu)|=f(\lambda,\mu)\,g(\lambda,\mu),$$

and the inverse matrix is as follows

$$(F(\lambda,\mu))^{-1}=\begin{pmatrix} \frac{1}{f(\lambda,\mu)}+\frac{1}{g(\lambda,\mu)} & \frac{-1}{g(\lambda,\mu)}\\ \frac{-1}{g(\lambda,\mu)} & \frac{1}{g(\lambda,\mu)}\end{pmatrix}.$$

The matrix $B_1(\lambda)$ *is of the form*

$$B_1(\lambda)=\left|1-b_1e^{i\lambda}\right|^2\begin{pmatrix} 2+b_2^2 & -1 & -b_2 & 0\\ -1 & 1 & 0 & 0\\ -b_2 & 0 & 2+b_2^2 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}.$$

The inverse matrix $B_1^{-1}(\lambda)$ *equals to*

$$\frac{1}{A\,|1-b_1e^{i\lambda}|^2}\begin{pmatrix} 1+b_2^2 & 1+b_2^2 & b_2 & b_2\\ 1+b_2^2 & 2+2b_2^2+b_2^4 & b_2 & b_2\\ b_2 & b_2 & 1+b_2^2 & 1+b_2^2\\ b_2 & b_2 & 1+b_2^2 & 2+2b_2^2+b_2^4\end{pmatrix},$$

where $A=1+b_2^2+b_2^4$*. Let*

$$\vec{a}(0,0)=(\alpha,\,\beta)^{\top},\ \vec{a}(0,1)=(\gamma,\,\delta)^{\top}.$$

Then we will have $\vec{\mathbf{c}}_1(\lambda)^{\top}=(\vec{c}_0(\lambda)^{\top},\vec{c}_1(\lambda)^{\top})$*, where*

$$\vec{c}_0(\lambda)=\frac{1}{A\,|1-b_1e^{i\lambda}|^2}\big((1+b_2^2)(\alpha+\beta)+b_2(\gamma+\delta),(1+b_2^2)(\alpha+\beta)+b_2(\gamma+\delta)$$

$$+(1+b_2^2+b_2^4)\beta\big)^{\top},$$

$$\vec{c}_1(\lambda)=\frac{1}{A\,|1-b_1e^{i\lambda}|^2}\big(b_2(\alpha+\beta)+(1+b_2^2)(\gamma+\delta),\ b_2(\alpha+\beta)+(1+b_2^2)(\gamma+\delta)$$

$$+(1+b_2^2+b_2^4)\delta)^\top.$$

Thus, the spectral characteristic of the optimal estimate is calculated by the formula

$$(h_1(F))^\top = \frac{b_2}{A\left|1-b_1 e^{i\lambda}\right|^2}(h(\mu),\,0)^\top.$$

where

$$h(\mu) = ((1+b_2^2)(\alpha+\beta)+b_2(\gamma+\delta))e^{-i\mu}+(b_2(\alpha+\beta)+(1+b_2^2)(\gamma+\delta))e^{2i\mu},$$

The mean-square error is of the form

$$\Delta_1(F) = \frac{((\alpha+\beta)^2+(\gamma+\delta)^2)(1+b_2^2)+2(\alpha+\beta)(\gamma+\delta)b_2}{2\pi A}\int_{-\pi}^{\pi}\frac{d\lambda}{\left|1-b_1 e^{i\lambda}\right|^2}.$$

2 Minimax method of interpolation

Derived theorem 1 and corollaries can be used only in the case of spectral certainty, when spectral densities of fields are exactly known. However, in practice, this case is not common, we do not have the exact values of spectral densities. If there is given a class of admissible spectral densities where spectral densities of the fields belong to, minimax method can be used to estimate the functionals. This method allows us to find estimates that minimize the maximum values of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities.

Definition 1. *For a given class of spectral densities $D=D_F\times D_G$ the spectral densities $F^0(\lambda,\mu)\in D_F$, $G^0(\lambda,\mu)\in D_G$ are called least favorable in the class D for the optimal linear interpolation of the functional $A_K\vec{\xi}$ if the following relation holds true*

$$\Delta\left(F^0,G^0\right) = \Delta\left(h\left(F^0,G^0\right);F^0,G^0\right) = \max_{(F,G)\in D_F\times D_G}\Delta\left(h\left(F,G\right);F,G\right).$$

Definition 2. *For a given class of spectral densities $D=D_F\times D_G$ the spectral characteristic $h^0(\lambda,\mu)$ of the optimal linear interpolation of the functional $A_K\vec{\xi}$ is called minimax-robust if there are satisfied conditions*

$$h^0(\lambda,\mu)\in H_D = \bigcap_{(F,G)\in D_F\times D_G} L_2^{K-}(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta\left(h;F,G\right) = \max_{(F,G)\in D}\Delta\left(h^0;F,G\right).$$

The least favorable spectral densities $F^0(\lambda,\mu)$, $G^0(\lambda,\mu)$ and the minimax spectral characteristic $h^0=h(F^0,G^0)$ form a saddle point of the function $\Delta\left(h;F,G\right)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta\left(h^0;F,G\right)\le\Delta\left(h^0;F^0,G^0\right)\le\Delta\left(h;F^0,G^0\right),\quad \forall h\in H_D,\forall F\in D_F,\forall G\in D_G,$$

hold true if $h^0=h(F^0,G^0)$ and $h(F^0,G^0)\in H_D$, where (F^0,G^0) is a solution to the constrained optimization problem

$$\sup_{(F,G)\in D_F\times D_G}\Delta\left(h(F^0,G^0);F,G\right) = \Delta\left(h(F^0,G^0);F^0,G^0\right), \tag{12}$$

$$\Delta\left(h\left(F^0,G^0\right);F,G\right)$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((A_K(\lambda,\mu))^\top G^0(\lambda,\mu)+(C_K^0(\lambda,\mu))^\top)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}F(\lambda,\mu)$$
$$\times(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}((A_K(\lambda,\mu))^\top G^0(\lambda,\mu)+(C_K^0(\lambda,\mu))^\top)^* d\lambda d\mu$$
$$+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}((A_K(\lambda,\mu))^\top F^0(\lambda,\mu)-(C_K^0(\lambda,\mu))^\top)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}G(\lambda,\mu)$$
$$\times(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}((A_K(\lambda,\mu))^\top F^0(\lambda,\mu)-(C_K^0(\lambda,\mu))^\top)^* d\lambda d\mu.$$

From the introduced definitions and formulas derived in previous section we can obtain the following statements.

Lemma 1. *Spectral densities $F^0(\lambda,\mu)\in D_F$, $G^0(\lambda,\mu)\in D_G$ satisfying the minimality condition (1) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear interpolation of the functional $A_K\vec{\xi}$ if the Fourier coefficients of the functions*

$$((F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1})^\top,(F^0(\lambda,\mu)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1})^\top,$$
$$(F^0(\lambda,\mu)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}G^0(\lambda,\mu))^\top$$

determine

$$C_K^0(\lambda,\mu)=\sum_{(k,j)\in K}\vec{c}^{\,0}(k,j)e^{i(k\lambda+j\mu)}$$

which determine a solution to the constrained optimization problem (12). The minimax spectral characteristic $h^0=h(F^0,G^0)$ can be calculated by the formula (3) if $h(F^0,G^0)\in H_D$.

Corollary 3. *Spectral densities $F^0(\lambda,\mu)\in D_F$, $G^0(\lambda,\mu)\in D_G$ satisfying the minimality condition (1) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear interpolation of the functional $A_N\vec{\xi}$ if the Fourier coefficients of the functions*

$$((F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1})^\top,(F^0(\lambda,\mu)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1})^\top,$$
$$(F^0(\lambda,\mu)(F^0(\lambda,\mu)+G^0(\lambda,\mu))^{-1}G^0(\lambda,\mu))^\top$$

determine the matrices $B_N^0(\lambda),D_N^0(\lambda),R_N^0(\lambda)$, which determine a solution to the constrained optimization problem

$$\max_{(F,G)\in D_F\times D_G}\frac{1}{2\pi}\int_{-\pi}^{\pi}(<\vec{a}_N(\lambda),R_N(\lambda)\vec{a}_N(\lambda)>+<B_N^{-1}(\lambda)D_N(\lambda)\vec{a}_N(\lambda),D_N(\lambda)\vec{a}_N(\lambda)>)d\lambda$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}(<\vec{a}_N(\lambda),R_N^0(\lambda)\vec{a}_N(\lambda)>+<B_N^{0-1}(\lambda)D_N^0(\lambda)\vec{a}_N(\lambda),D_N^0(\lambda)\vec{a}_N(\lambda)>)d\lambda. \quad (13)$$

The minimax spectral characteristic $h^0=h(F^0,G^0)$ is determined by the formula (3) if $h(F^0,G^0)\in H_D$.

Corollary 4. *Let the spectral density $F^0(\lambda,\mu)\in D_F$ satisfy the minimality condition (9). The spectral density $F^0(\lambda,\mu)\in D_F$ is the least favorable in the class D_F for the optimal linear interpolation of the functional $A_N\vec{\xi}$ from the observation of the field $\vec{\xi}(k,j)$ at points $(k,j)\in Z^2\setminus K$, $K=\{(k,j):k\in Z,0\le j\le N\}$ if the Fourier coefficients of the function $((F^0(\lambda,\mu))^{-1})^\top$ determine the matrix $B_N^0(\lambda)$ which determines a solution to the constrain optimization problem*

$$\max_{F\in D_F}\frac{1}{2\pi}\int_{-\pi}^{\pi}<\vec{a}_N(\lambda),B_N^{-1}(\lambda)\vec{a}_N(\lambda)>d\lambda=\frac{1}{2\pi}\int_{-\pi}^{\pi}<\vec{a}_N(\lambda),B_N^{0-1}(\lambda)\vec{a}_N(\lambda)>d\lambda. \quad (14)$$

The minimax spectral characteristic $h^0=h(F^0)$ is determined by the formula (10) if $h(F^0)\in H_{D_F}$.

The constrained optimization problem (12) is equivalent to the unconstrained optimization problem [15]:

$$\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)\,|D_F\times D_G\,) \to \inf, \tag{15}$$

where $\delta((F,G)|D_F\times D_G)$ is the indicator function of the set $D=D_F\times D_G$. Solution of the problem (15) is characterized by the condition $0\in\partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0) [16]. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities [15].

Lemma 2. *Let (F^0,G^0) be a solution to the optimization problem* (15). *The spectral densities $F^0(\lambda,\mu)$, $G^0(\lambda,\mu)$ are the least favorable in the class $D=D_F\times D_G$ and the spectral characteristic $h^0=h(F^0,G^0)$ is the minimax of the optimal linear estimate of the functional $A_K\vec{\xi}$ if $h(F^0,G^0)\in H_D$.*

3 Least favorable spectral densities in the class $D=D_0\times D_\varepsilon$

Consider the problem of the mean-square optimal interpolation of the functional $A_N\vec{\xi}$ in the case when spectral densities of fields belong to the class of admissible spectral densities $D=D_0^k\times D_\varepsilon^k, k=1,2,3,4$, where

$$D_0^1=\left\{F(\lambda,\mu)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\,F(\lambda,\mu)d\mu=p(\lambda)\right.\right\},$$

$$D_\varepsilon^1=\left\{G(\lambda,\mu)\left|\mathrm{Tr}\,G(\lambda,\mu)=(1-\varepsilon)\mathrm{Tr}\,G_1(\lambda,\mu)+\varepsilon\mathrm{Tr}\,W(\lambda,\mu),\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\,G(\lambda,\mu)d\mu=q(\lambda)\right.\right\};$$

$$D_0^2=\left\{F(\lambda,\mu)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f_{kk}(\lambda,\mu)d\mu=p_k(\lambda),k=\overline{1,T}\right.\right\},$$

$$D_\varepsilon^2=\left\{G(\lambda,\mu)\left|g_{kk}(\lambda,\mu)=(1-\varepsilon)g_{kk}^1(\lambda,\mu)+\varepsilon w_{kk}(\lambda,\mu),\right.\right.$$

$$\left.\frac{1}{2\pi}\int_{-\pi}^{\pi}g_{kk}(\lambda,\mu)d\mu=q_k(\lambda),k=\overline{1,T}\right\};$$

$$D_0^3=\left\{F(\lambda,\mu)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle B_1,F(\lambda,\mu)\rangle\,d\mu=p(\lambda)\right.\right\},$$

$$D_\varepsilon^3=\left\{G(\lambda,\mu)\left|\langle B_2,G(\lambda,\mu)\rangle=(1-\varepsilon)\langle B_2,G_1(\lambda,\mu)\rangle+\varepsilon\langle B_2,W(\lambda,\mu)\rangle,\right.\right.$$

$$\left.\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle B_2,G(\lambda,\mu)\rangle\,d\mu=q(\lambda)\right\};$$

$$D_0^4=\left\{F(\lambda,\mu)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\lambda,\mu)d\mu=P(\lambda)\right.\right\},$$

$$D_\varepsilon^4=\left\{G(\lambda,\mu)\left|G(\lambda,\mu)=(1-\varepsilon)G_1(\lambda,\mu)+\varepsilon W(\lambda,\mu),\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\lambda,\mu)d\mu=Q(\lambda)\right.\right\},$$

where $G_1(\lambda,\mu)$ is a known and fixed spectral density matrix while $W(\lambda,\mu)$ is an unknown spectral density matrix, $p(\lambda),q(\lambda),p_k(\lambda),q_k(\lambda),k=\overline{1,T}$ are given functions, $P(\lambda)$, $Q(\lambda)$, $B_1(\lambda)$, $B_2(\lambda)$ are given matrices.

From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair $D_0^1 \times D_\varepsilon^1$ we have equations

$$((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)$$
$$= \alpha^2(\lambda)(F^0(\lambda,\mu) + G^0(\lambda,\mu))^2, \tag{16}$$
$$((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)$$
$$= (\beta^2(\lambda) + \gamma(\lambda,\mu))(F^0(\lambda,\mu) + G^0(\lambda,\mu))^2, \tag{17}$$

where $\gamma(\lambda,\mu) \le 0$ and $\gamma(\lambda,\mu) = 0$ if $\mathrm{Tr}\, F^0(\lambda,\mu) > (1-\varepsilon)\mathrm{Tr}\, G_1(\lambda,\mu)$.

For the second pair $D_0^2 \times D_\varepsilon^2$ we have equations

$$((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)$$
$$= (F^0(\lambda,\mu) + G^0(\lambda,\mu)) \left\{\alpha_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^T (F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{18}$$
$$((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)$$
$$= (F^0(\lambda,\mu) + G^0(\lambda,\mu)) \left\{(\beta_k^2(\lambda) + \gamma_k(\lambda,\mu))\delta_{kl}\right\}_{k,l=1}^T (F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{19}$$

where $\gamma_k(\lambda,\mu) \le 0$ and $\gamma_k(\lambda,\mu) = 0$ if $g_{kk}^0(\lambda,\mu) > (1-\varepsilon)g_{kk}^1(\lambda,\mu)$.

For the third pair $D_0^3 \times D_\varepsilon^3$ we have equations

$$((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)$$
$$= \alpha^2(\lambda)(F^0(\lambda,\mu) + G^0(\lambda,\mu))B_1^\top (F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{20}$$
$$((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)$$
$$= (\beta^2(\lambda) + \gamma'(\lambda,\mu))(F^0(\lambda,\mu) + G^0(\lambda,\mu))B_2^\top (F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{21}$$

where $\gamma'(\lambda,\mu) \le 0$ and $\gamma'(\lambda,\mu) = 0$ if $\langle B_2, G^0(\lambda,\mu)\rangle > (1-\varepsilon)\langle B_2, G_1(\lambda,\mu)\rangle$.

For the fourth pair $D_0^4 \times D_\varepsilon^4$ we have equations

$$((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top G^0(\lambda,\mu) + (C_N^0(\lambda,\mu))^\top)$$
$$= (F^0(\lambda,\mu) + G^0(\lambda,\mu))\vec{\alpha}(\lambda) \cdot \vec{\alpha}(\lambda)^*(F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{22}$$
$$((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)^*((A_N(\lambda,\mu))^\top F^0(\lambda,\mu) - (C_N^0(\lambda,\mu))^\top)$$
$$= (F^0(\lambda,\mu) + G^0(\lambda,\mu))(\vec{\beta}(\lambda) \cdot \vec{\beta}(\lambda)^* + \Gamma(\lambda,\mu))(F^0(\lambda,\mu) + G^0(\lambda,\mu)), \tag{23}$$

where $\Gamma(\lambda,\mu) \le 0$ and $\Gamma(\lambda,\mu) = 0$ if $G^0(\lambda,\mu) > (1-\varepsilon)G_1(\lambda,\mu)$.

Thus, the following statements hold true.

Theorem 2. *Let the minimality condition* (1) *hold true. The least favorable spectral densities* $F^0(\lambda,\mu)$, $G^0(\lambda,\mu)$ *in the classes* $D_0 \times D_\varepsilon$ *for the optimal linear interpolation of the functional* $A_N\vec{\xi}$ *are determined by relations* (16), (17) *for the first pair* $D_0^1 \times D_\varepsilon^1$ *of sets of admissible spectral densities;* (18), (19) *for the second pair* $D_0^2 \times D_\varepsilon^2$ *of sets of admissible spectral densities;* (20), (21) *for the third pair* $D_0^3 \times D_\varepsilon^3$ *of sets of admissible spectral densities;* (22), (23) *for the fourth pair* $D_0^4 \times D_\varepsilon^4$ *of sets of admissible spectral densities; constrained optimization problem* (13) *and restrictions on densities from the corresponding classes* $D_0 \times D_\varepsilon$. *The minimax-robust spectral characteristic of the optimal estimate of the functional* $A_N\vec{\xi}$ *is determined by the formula* (3).

Corollary 5. *Let the minimality condition* (9) *hold true. The least favorable spectral densities* $F^0(\lambda,\mu)$ *in the classes* D_0^k, $k=1,2,3,4$, *for the optimal linear interpolation of the functional* $A_N\vec{\xi}$, *which depends on the unknown values of the field* $\vec{\xi}(k,j)$ *based on observations of the field* $\vec{\xi}(k,j)$ *at points* $(k,j)\in Z^2\setminus K,\ K=\{(k,j):k\in Z,0\leq j\leq N\}$, *are determined by the following equations, respectively,*

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=\alpha^2(\lambda)(F^0(\lambda,\mu))^2, \tag{24}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=F^0(\lambda,\mu)\left\{\alpha_k^2(\lambda)\delta_{kl}\right\}_{k,l=1}^T F^0(\lambda,\mu), \tag{25}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=\alpha^2(\lambda)F^0(\lambda,\mu)(B_2)^\top F^0(\lambda,\mu), \tag{26}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=F^0(\lambda,\mu)\vec{\alpha}(\lambda)\cdot\vec{\alpha}(\lambda)^*F^0(\lambda,\mu), \tag{27}$$

constrained optimization problem (14) *and restrictions on densities from the corresponding classes* D_0^k, $k=1,2,3,4$. *The minimax spectral characteristic of the optimal estimate of the functional* $A_N\vec{\xi}$ *is determined by the formula* (10).

Corollary 6. *Let the minimality condition* (9) *hold true. The least favorable spectral densities* $F^0(\lambda,\mu)$ *in the classes* D_ε^k, $k=1,2,3,4$, *for the optimal linear interpolation of the functional* $A_N\vec{\xi}$, *which depends on the unknown values of the field* $\vec{\xi}(k,j)$ *based on observations of the field* $\vec{\xi}(k,j)$ *at points* $(k,j)\in Z^2\setminus K,\ K=\{(k,j):k\in Z,0\leq j\leq N\}$, *are determined by the following equations, respectively,*

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=(\beta^2(\lambda)+\gamma(\lambda,\mu))(F^0(\lambda,\mu))^2, \tag{28}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=F^0(\lambda,\mu)\left\{(\beta_k^2(\lambda)+\gamma_k(\lambda,\mu))\delta_{kl}\right\}_{k,l=1}^T F^0(\lambda,\mu), \tag{29}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=(\beta^2(\lambda)+\gamma'(\lambda,\mu))F^0(\lambda,\mu)(B_2)^\top F^0(\lambda,\mu), \tag{30}$$

$$((C_N^0(\lambda,\mu))^\top)^*\cdot(C_N^0(\lambda,\mu))^\top=F^0(\lambda,\mu)(\vec{\beta}(\lambda)\cdot\vec{\beta}(\lambda)^*+\Gamma(\lambda,\mu))F^0(\lambda,\mu), \tag{31}$$

constrained optimization problem (14) *and restrictions on densities from the corresponding classes* D_ε^k, $k=1,2,3,4$. *The minimax spectral characteristic of the optimal estimate of the functional* $A_N\vec{\xi}$ *is determined by the formula* (10).

CONCLUSIONS

In this article we propose methods of the mean-square optimal linear estimation of the functionals which depend on the unknown values of the multidimensional homogeneous random field based on observed data of the field with noise. Under condition of spectral certainty where the spectral densities of the fields are exactly known we derive formulas for calculating the spectral characteristics and the mean-square errors of the estimates of the functionals. Analogous results are derived for the case of observations of the field without noise. In the case of spectral uncertainty, where the spectral densities are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functionals are proposed for some specific classes of admissible spectral densities.

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Серед сучасних напрямків теорії стохастичних процесів, таких як узагальнені стохастичні процеси, властивості функціоналів від випадкових процесів, статистичні методи стохастичних процесів та інші, важливим є напрямок, зосереджений на проблемі оцінювання невідомих значень випадкових процесів. Особливої актуальності в останні роки набуває проблема оцінювання значень процесів в умовах невизначеності. Тому розробка методів оцінювання є одним із основних завдань сучасної теорії стохастичних процесів. У даній статті ми розглядаємо проблему оптимальної лінійної інтерполяції функціоналу, що залежать від невідомих значень векторного стаціонарного випадкового поля за спостереженнями поля з шумом. Ми застосовуємо класичний підхід для виведення формул для обчислення значень середньоквадратичної похибки та спектральної характеристики оптимальної лінійної оцінки функціоналу. Цей підхід базується на припущенні, що спектральні щільності стаціонарних полів точно відомі. Однак на практиці повна інформація про спектральні щільності в більшості випадків неможлива. Щоб подолати цю проблему, знаходять параметричні або непараметричні оцінки невідомих спектральних щільностей або вибирають щільності з інших міркувань, а потім застосовують класичний метод оцінювання за умови, що оцінена або вибрана щільність є істинною. Такий підхід може призвести до значного зростання величини похибки оцінювання. Це дає стимул до пошуку оцінок, оптимальних для всіх щільностей з певного класу допустимих спектральних щільностей. Такі оцінки називаються мінімаксними, оскільки вони мінімізують максимальне значення похибки оцінювання. Тому у випадку спектральної невизначеності ми використовуємо мінімаксний підхід і пропонуємо формули, які визначають найменш сприятливі спектральні щільності та мінімаксні спектральні характеристики оптимальних оцінок функціоналу для деяких класів допустимих спектральних щільностей.