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INITIAL BOUNDARY VALUE PROBLEM FOR INTEGRO-DIFFERENTIAL SYSTEMS WITH VARIABLE EXPONENTS OF NONLINEARITY

In this paper, we investigate the problem for a fourth order evolutionary systems with a shift. The existence of the weak solution of the problem is proved.

Key words and phrases: partial differential equation, parabolic equation, weak solution, variable exponent of nonlinearity.

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INTRODUCTION

Let $N, n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$, $Q_{0,\tau} := \Omega \times (0, \tau)$, $\Sigma_{0,\tau} := \partial\Omega \times (0, \tau)$, $\tau \in (0, T]$.

We seek a vector-valued function $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_N) : Q_{0,T} \rightarrow \mathbb{R}^N$ such that

$$\begin{aligned} & \tilde{u}_{k,t} + \alpha \Delta^2 (\tilde{u}_k + b_k) + \Delta \left(c_k(x, t) \left| \tilde{u} + b(x, t) \right|^{\gamma(x,t)-2} (\tilde{u}_k + b_k(x, t)) \right) - \\ & - \sum_{i=1}^n \left(a_{ik}(x, t) \left| \tilde{u}_{x_i} + b_{x_i}(x, t) \right|^{p(x,t)-2} (\tilde{u}_k + b_k(x, t))_{x_i} \right)_{x_i} + \\ & + \left(\mathbf{N}(\tilde{u} + b) \right)_k(x, t) + \varphi_k \left(\left(\mathbf{E}(\tilde{u} + b) \right)_k(x, t) \right) = F_k(x, t), \quad (x, t) \in Q_{0,T}, \end{aligned} \quad (1)$$

$$\tilde{u}|_{\Sigma_{0,T}} = \Delta \tilde{u}|_{\Sigma_{0,T}} = 0, \quad (2)$$

$$\tilde{u}(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

where $\alpha > 0$ is some number; $\Delta \equiv \Delta^1$ is the Laplacian; $\Delta^{s+1} := \Delta(\Delta^s)$ for $s \in \mathbb{N}$;

$$(\mathbf{N}z)_k(x, t) \equiv (N(t)z(t))_k(x) := g_k(x, t) |z(x, t)|^{q(x,t)-2} z_k(x, t), \quad (4)$$

$$(\mathbf{E}z)_k(x, t) \equiv (E(t)z(t))_k(x) := \int_{\Omega} \mathfrak{Z}_k(x, t, y) z_k(y, t) dy, \quad z = (z_1, \dots, z_N), \quad (x, t) \in Q_{0,T}; \quad (5)$$

$$\mathbf{N}z := ((\mathbf{N}z)_1, \dots, (\mathbf{N}z)_N); \quad \mathbf{E}z := ((\mathbf{E}z)_1, \dots, (\mathbf{E}z)_N); \quad (6)$$

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$c_k, a_{ik}, g_k, \varphi_k, \mathfrak{F}_k, F_k, u_0, b = (b_1, \dots, b_N)$ are some functions, $i = \overline{1, n}$, $k = \overline{1, N}$; $\gamma = \gamma(x, t)$, $p = p(x, t)$, and $q = q(x, t)$ are the variable exponents of the nonlinearity.

In Theorem 1 below we prove the existence of the weak solution to problem (1)-(3). The corresponding to (1) equation with the second derivative with respect to the time variable, under the condition $b_k \equiv 0$ and without integral terms, was considered by N. Protsakh in [2]. The system of second-order equation and fourth-order semilinear equation with variable exponents of nonlinearity and without integral terms was investigated by T. Bokalo in [3]. When $b \equiv 0$ and when γ, p, q do not depend on t the problem (1)-(3) was investigated by O. Buhrii and N. Buhrii in [4]. Various problem for the equations with the variable exponents of the nonlinearity were considered in [5], [6], [7], [8], [9], [10]. The problem for the system (1) with $b \neq 0$ and an integral term present there and variable exponents of nonlinearity was considered for the first time.

1 NOTATION AND STATEMENT OF PROBLEM

We use the notation from paper [4]. Additionally, suppose that $\text{Lip}(\mathbb{R})$ be a set of the Lipschitz functions on \mathbb{R} , for some $m \in \mathbb{N}$ and some measurable set $\mathcal{O} \subset \mathbb{R}^m$ we also put $L^0(\mathcal{O}) := \{v : \mathcal{O} \rightarrow \mathbb{R} \mid v \text{ is a measurable}\}$, and $\mathcal{B}_+(\mathcal{O}) := \{q \in L^\infty(\mathcal{O}) \mid \text{ess inf}_{y \in \mathcal{O}} q(y) > 0\}$. For any function $q \in \mathcal{B}_+(\mathcal{O})$ let us denote

$$q_0 := \text{ess inf}_{y \in \mathcal{O}} q(y), \quad q^0 := \text{ess sup}_{y \in \mathcal{O}} q(y), \quad S_q(s) := \max\{s^{q_0}, s^{q^0}\}, \quad s \geq 0, \quad (7)$$

$$q'(y) := \frac{q(y)}{q(y) - 1} \text{ for almost every (a.e.) } y \in \mathcal{O}. \quad (8)$$

Suppose that

$$\mathcal{P}^{\log}(\mathcal{O}) := \{q \in L^\infty(\mathcal{O}) \mid q \text{ - globally log-Holder continuous function, } q_0 \geq 1\}, \quad (9)$$

$$H := [L^2(\Omega)]^N, \quad V_s := [H_0^s(\Omega)]^N, \quad (10)$$

$$\mathcal{W}_s := \{v \in H^{2s}(\Omega) \mid v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = \dots = \Delta^{s-1}v|_{\partial\Omega} = 0\}, \quad \mathcal{W}_s^* := [\mathcal{W}_s]^*, \quad (11)$$

$$Z := \mathcal{W}_1, \quad X(t) := W_0^{1,p(x,t)}(\Omega), \quad Y(t) := L^{q(x,t)}(\Omega), \quad (12)$$

$$V(t) := Z^N \cap [X(t)]^N \cap [Y(t)]^N \cap H, \quad (13)$$

$$X^*(t) := [X(t)]^*, \quad Y^*(t) := [Y(t)]^*, \quad H^* \simeq H, \quad V^*(t) := [V(t)]^*, \quad t \in [0, T], \quad (14)$$

$$U(Q_{0,T}) := \{u : (0, T) \rightarrow V(t) \mid D^\alpha u \in [L^2(Q_{0,T})]^N \text{ when } |\alpha| = 2, \\ u_{x_1}, \dots, u_{x_n} \in [L^{p(x,t)}(Q_{0,T})]^N, \quad u \in [L^{q(x,t)}(Q_{0,T})]^N \cap [L^2(Q_{0,T})]^N\}, \quad (15)$$

$$W(Q_{0,T}) := \{w \in U(Q_{0,T}) \mid w_t \in [U(Q_{0,T})]^*\}. \quad (16)$$

Note that $W(Q_{0,T}) \subset C([0, T]; H)$. Let's assume that the following conditions are satisfied:

(P): $\gamma, p, q \in \mathcal{P}^{\log}(Q_{0,T})$;

(Z): $\alpha > 0$, $\gamma^0 \leq 2$; $r_0 = \min\{2, p_0, q_0\}$, $r^0 = \max\{2, p^0, q^0\}$, $s \in \mathbb{N}$,

$$s \geq \frac{1}{2} \max\left\{2, 1 + \frac{n(p^0 - 2)}{2p^0}, \frac{n(q^0 - 2)}{2q^0}\right\};$$

- (A): $a_{ik} \in L^0(Q_{0,T})$, $0 < a_0 \leq a_{ik}(x, t) \leq a^0 < +\infty$ for a.e. $(x, t) \in Q_{0,T}$,
 where $i = \overline{1, n}$, $k = \overline{1, N}$;
- (C): $c_k \in L^0(Q_{0,T})$, $|c_k(x, t)| \leq c^0 < +\infty$ for a.e. $(x, t) \in Q_{0,T}$, where $k = \overline{1, N}$;
- (G): $g_k \in L^0(Q_{0,T})$, $0 < g_0 \leq g_k(x, t) \leq g^0 < +\infty$ for a.e. $(x, t) \in Q_{0,T}$, where $k = \overline{1, N}$;
- (Φ): $\varphi_k \in \text{Lip}(\mathbb{R})$, $|\varphi_k(\xi)| \leq \varphi^0 |\xi|$ for a.e. $\xi \in \mathbb{R}$, where $0 \leq \varphi^0 < +\infty$, $k = \overline{1, N}$;
- (E): $\mathfrak{Z}_k \in L^0(Q_{0,T} \times \Omega)$, $|\mathfrak{Z}_k(x, t, y)| \leq \mathfrak{Z}^0 < +\infty$ for a.e. $(x, t, y) \in Q_{0,T} \times \Omega$,
 where $k = \overline{1, N}$;
- (FD): $F \in L^2(0, T; H)$;
- (UD): $u_0 \in H$;
- (WD): $b \in U(Q_{0,T})$.

Also we introduce the following notations:

$$(z, v)_\Omega := \begin{cases} \int_{\Omega} (z(x), v(x))_{\mathbb{R}^N} dx, & z = (u_1, \dots, u_N) : \Omega \rightarrow \mathbb{R}^N, \\ & v = (v_1, \dots, v_N) : \Omega \rightarrow \mathbb{R}^N, \\ \int_{\Omega} z(x)v(x) dx, & z, v : \Omega \rightarrow \mathbb{R}; \end{cases} \quad (17)$$

$$\begin{aligned} \langle S(t)z, w \rangle_{V(t)} &:= \int_{\Omega} \sum_{k=1}^N \left[\alpha \Delta z_k \Delta w_k + c_k(x, t) |z|^{\gamma(x,t)-2} z_k \Delta w_k + \right. \\ &+ \sum_{i=1}^n a_{ik}(x, t) |z_{x_i}|^{p(x,t)-2} z_{k,x_i} w_{k,x_i} + \varphi_k \left((\mathbf{E}z)_k \right) w_k \left. \right] dx + \\ &+ \left((\mathbf{N}z)(t), w \right)_{\Omega}, \quad z, w \in V(t), \quad t \in (0, T); \end{aligned} \quad (18)$$

$$\langle \mathbf{S}u, v \rangle_{U(Q_{0,T})} := \int_0^T \langle S(t)u(t), v(t) \rangle_{V(t)} dt, \quad u, v \in U(Q_{0,T}). \quad (19)$$

Let us give the definition of the solution to our problem.

Definition 1. Function $\tilde{u} : Q_{0,T} \rightarrow \mathbb{R}^N$ is called a weak solution to problem (1)-(3), if

- 1) $\tilde{u} \in W(Q_{0,T})$;
- 2) function \tilde{u} satisfies the equality

$$\begin{aligned} \int_{Q_{0,T}} \sum_{k=1}^N \left[-\tilde{u}_k v_{k,t} + \alpha \Delta (\tilde{u}_k + b_k) \Delta v_k + c_k(x, t) |\tilde{u} + b|^{\gamma(x,t)-2} (\tilde{u}_k + b_k) \Delta v_k + \right. \\ \left. + \sum_{i=1}^n a_{ik}(x, t) |\tilde{u}_{x_i} + b_{x_i}|^{p(x,t)-2} (\tilde{u}_{k,x_i} + b_{k,x_i}) v_{k,x_i} + \right. \\ \left. + \left(\mathbf{N}(\tilde{u} + b) \right)_k v_k + \varphi_k \left(\left(\mathbf{E}(\tilde{u} + b) \right)_k \right) v_k \right] dx dt = \int_{Q_{0,T}} \sum_{k=1}^N F_k v_k dx dt \end{aligned} \quad (20)$$

for all test functions $v \in U(Q_{0,T})$, i.e., in the sense of the spaces $[U(Q_{0,T})]^*$ and $[D^*(Q_{0,T})]^N$ it holds the equality

$$\tilde{u}_t + \mathbf{S}(\tilde{u} + b) = F; \quad (21)$$

- 3) \tilde{u} satisfies condition (3) in the sense of the space $C([0, T]; H)$.

The main result of this paper is the following.

Theorem 1. Let conditions (P)-(WD) hold, the constant s is taken from condition (Z), and $\partial\Omega \in C^{2s}$. Then problem (1)-(3) has a weak solution.

2 AUXILIARY STATEMENTS

Similarly to Lemma 3.26 [4, p. 875] we obtain the following.

Lemma 1. *If condition **(E)** is satisfied, then $\mathbf{E} : L^2(0, T; H) \rightarrow L^2(0, T; H)$ is a linear, bounded, continuous operator and it satisfies the estimate*

$$\|\mathbf{E}u; L^2(0, T; H)\| \leq C_1 \|u; L^2(0, T; H)\|, \quad u \in L^2(0, T; H), \quad (22)$$

where $C_1 > 0$ – constant, which does not depend on u .

We will use the additional notations. Suppose that,

$$V_- := Z^N \cap [W_0^{1,p_0}(\Omega)]^N \cap [L^{q_0}(\Omega)]^N \cap H, \quad V_+ := Z^N \cap [W_0^{1,p^0}(\Omega)]^N \cap [L^{q^0}(\Omega)]^N \cap H. \quad (23)$$

Note that the space \mathcal{W}_s from (11) is a Hilbert space, it is also reflexive, and

$$\mathcal{W}_s \circlearrowleft H^{2s}(\Omega), \quad \mathcal{W}_s \circlearrowright L^2(\Omega) \circlearrowright \mathcal{W}_s^*. \quad (24)$$

From the choice of the number s and space $V(t)$ we obtain that

$$[\mathcal{W}_s]^N \circlearrowright V(t) \circlearrowright H \cong H^* \circlearrowright V^*(t) \circlearrowright [\mathcal{W}_s^*]^N. \quad (25)$$

Let r_0, r^0 be taken from **(Z)**. Then, similarly to Lemma 2 [11, p. 46], we have

$$L^{r^0}(0, T; V_+) \circlearrowright U(Q_{0,T}) \circlearrowright L^{r_0}(0, T; V_-). \quad (26)$$

Similarly to (25) we obtain the following expression

$$L^{r^0}(0, T; [\mathcal{W}_s]^N) \circlearrowright U(Q_{0,T}) \circlearrowright [L^2(Q_{0,T})]^N \circlearrowright [U(Q_{0,T})]^* \circlearrowright L^{\frac{r^0}{r_0-1}}(0, T; [\mathcal{W}_s^*]^N). \quad (27)$$

We will use the following facts.

Proposition 1 (see Lemma 1 [12], p. 32). *Let the condition **(P)** be satisfied, $g \in L^\infty(Q_{0,T})$, $z \in L^{q^0}(\Omega)$, $m \in \mathbb{N}$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $w^1, \dots, w^m \in L^{q^0}(\Omega)$, $w(x, \xi) = \sum_{l=1}^m \xi_l w^l(x)$. Then the function $I(t, \xi) = \int_\Omega g(x, t) |w(x, \xi)|^{q(x,t)-2} w(x, \xi) z(x) dx$, $t \in (0, T)$, $\xi \in \mathbb{R}^m$, satisfies the L^∞ -condition of Caratheodory.*

Proposition 2 (see Lemma 3.27 [4], p. 875). *Let $\mathfrak{z} \in L^\infty(Q_{0,T} \times \Omega)$, $\varphi \in \text{Lip}(\mathbb{R})$, $z \in L^2(\Omega)$, $m \in \mathbb{N}$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $w^1, \dots, w^m \in L^2(\Omega)$, $w(x, \xi) = \sum_{l=1}^m \xi_l w^l(x)$, $x \in \Omega$, \mathbf{E} taken from (5) and $k \in \{1, \dots, N\}$. Then the function $J(t, \xi) = \int_\Omega \varphi\left((\mathbf{E}w(\cdot, \xi))_k(x, t)\right) z(x) dx$, $t \in (0, T)$, $\xi \in \mathbb{R}^m$, satisfies L^∞ -condition of Caratheodory.*

Lemma 2. *Let $\alpha > 0$, conditions **(P)**-**(E)**, **(WD)** and notations (18) are satisfied. Let also $\{w^j\}_{j \in \mathbb{N}} \subset V_+$, $m \in \mathbb{N}$, $L = (L_{11}, L_{21}, \dots, L_{m1}, \dots, L_{1N}, L_{2N}, \dots, L_{mN})$, where*

$$L_{\mu k}(t, \xi) = \langle (S(t)(z + b))_k, w^\mu \rangle, \quad k = \overline{1, N}, \quad \mu = \overline{1, m}, \quad t \in (0, T), \quad (28)$$

$\xi = (\xi_{11}, \xi_{21}, \dots, \xi_{m1}, \dots, \xi_{1N}, \xi_{2N}, \dots, \xi_{mN})$, $z = (z_1, \dots, z_N)$, $z_k(x) = \sum_{\ell=1}^m \xi_{\ell k} w^\ell(x)$, $x \in \Omega$, $k = \overline{1, N}$. Then

$$(L(t, \xi), \xi)_{\mathbb{R}^{mN}} \geq \int_{\Omega} \left[\frac{\alpha}{4} |\Delta(z+b)|^2 + \frac{a_0}{2} \sum_{i=1}^n |z_{x_i} + b_{x_i}|^{p(x,t)} + \frac{g_0}{2} |z+b|^{q(x,t)} - C_2 |z+b|^2 \right] dx - C_3 \beta(t) - C_4, \quad t \in (0, T), \quad (29)$$

where the constants $C_2, C_3, C_4 > 0$ do not depend on z, ξ, t ,

$$\beta(t) = \int_{\Omega} \left[|\Delta b(x, t)|^2 + \sum_{i=1}^n |b_{x_i}(x, t)|^{p(x,t)} + |b(x, t)|^{q(x,t)} + |b(x, t)|^2 \right] dx. \quad (30)$$

Proof. It is clear that

$$(L(t, \xi), \xi)_{\mathbb{R}^{mN}} = \langle S(t)(z+b), z \rangle = \langle S(t)(z+b), z+b \rangle - \langle S(t)(z+b), b \rangle. \quad (31)$$

Using the generalized Young inequality (see [4, p. 863]) with the exponents $\gamma(x, t) > 1$ and $\gamma'(x, t) := \frac{\gamma(x,t)}{\gamma(x,t)-1} > 1$, we obtain the estimate

$$J_1 := \sum_{k=1}^N \left| c_k(x, t) |z|^{\gamma(x,t)-2} z_k \Delta z_k \right| \leq C_5 |z|^{\gamma(x,t)-1} |\Delta z| \leq \kappa_0 |\Delta z|^{\gamma(x,t)} + C_6(\kappa_0) |z|^{\gamma(x,t)}, \quad (32)$$

where $\kappa_0 > 0$ and the constant $C_6(\kappa_0) > 0$ does not depend on x, t, k, m .

At those points, where $\gamma(x, t) = 2$, we have $J_1 \leq \kappa_0 |\Delta z|^2 + C_7(\kappa_0) |z|^2$. At those points, where $\gamma(x, t) < 2$, we have $\frac{2}{\gamma} > 1$, $\frac{\frac{2}{\gamma}}{\frac{2}{\gamma}-1} = \frac{2}{2-\gamma} > 1$, and then

$$|\Delta z|^{\gamma(x,t)} \leq |\Delta z|^{\gamma(x,t) \cdot \frac{2}{\gamma(x,t)}} + C_8 \cdot 1^{\frac{2}{2-\gamma(x,t)}} = |\Delta z|^2 + C_8.$$

Similarly, we have $|z|^{\gamma(x,t)} \leq |z|^2 + C_9$. Generally, we obtain

$$J_1 \leq \varepsilon_1 |\Delta z|^2 + C_{10}(\varepsilon_1)(1 + |z|^2). \quad (33)$$

Using estimate (33) with $z+b$ instead of z , we get

$$\begin{aligned} \langle S(t)(z+b), z+b \rangle &\geq \int_{\Omega} \left[\frac{\alpha}{2} |\Delta(z+b)|^2 + \right. \\ &\left. + a_0 \sum_{i=1}^n |z_{x_i} + b_{x_i}|^{p(x,t)} + g_0 |z+b|^{q(x,t)} - C_{11} |z+b|^2 \right] dx - C_{12}. \end{aligned} \quad (34)$$

For the second term of (31), we obtain

$$\begin{aligned} \left| \langle S(t)(z+b), b \rangle \right| &= \left| \sum_{k=1}^N \int_{\Omega} \left[\alpha \Delta(z_k + b_k) \Delta b_k + \right. \right. \\ &\left. + c_k |z+b|^{\gamma(x,t)-2} (z_k + b_k) \Delta b_k + \sum_{i=1}^n a_{ik} |z_{x_i} + b_{x_i}|^{p(x,t)-2} (z_{k,x_i} + b_{k,x_i}) b_{k,x_i} + \right. \\ &\left. + g_k |z+b|^{q(x,t)-2} (z_k + b_k) b_k \right] dx + (\Phi(z+b), b)_{\Omega} \right|. \end{aligned} \quad (35)$$

Let's transform the expressions here. For the first term of (35), we get the estimate

$$\begin{aligned} & \left| \sum_{k=1}^N \alpha \Delta(z_k + b_k) \Delta b_k \right| \leq \sum_{k=1}^N \left| \alpha \Delta(z_k + b_k) \Delta b_k \right| \leq \\ & \leq \sum_{k=1}^N \left(\varepsilon_1 |\Delta(z_k + b_k)|^2 + C_{13}(\varepsilon_1) |\Delta b_k|^2 \right) = \varepsilon_1 |\Delta(z + b)|^2 + C_{14}(\varepsilon_1) |\Delta b|^2. \end{aligned} \quad (36)$$

We estimate the second term of (35) as follows:

$$\begin{aligned} I &:= \left| \sum_{k=1}^N c_k |z + b|^{\gamma(x,t)-2} (z_k + b_k) \Delta b_k \right| \leq \sum_{k=1}^N |c_k| \cdot |z + b|^{\gamma(x,t)-2} |z_k + b_k| \cdot |\Delta b_k| \leq \\ & \leq \sum_{k=1}^N |c_k| \cdot |z + b|^{\gamma(x,t)-1} \cdot |\Delta b_k| \leq c^0 \sum_{k=1}^N |z + b|^{\gamma(x,t)-1} \cdot |\Delta b_k| \leq \\ & \leq |z + b|^{\gamma(x,t)} + C_{15} \sum_{k=1}^N |\Delta b_k|^{\gamma(x,t)}, \end{aligned}$$

where the constant $C_{15} > 0$ does not depend on x, t, k, m . At those points, where $\gamma(x, t) = 2$, we have $I \leq |z + b|^2 + C_{16} |\Delta b|^2$. At those points, where $\gamma(x, t) < 2$, we get

$$|z + b|^{\gamma(x,t)} \leq |z + b|^{\gamma(x,t) \cdot \frac{2}{\gamma(x,t)}} + C_{17} \cdot 1^{\frac{2}{2-\gamma(x,t)}} = |z + b|^2 + C_{17}.$$

$$\sum_{k=1}^N |\Delta b_k|^{\gamma(x,t)} \leq \sum_{k=1}^N \left(|\Delta b_k|^2 + C_{18} \right) = |\Delta b|^2 + C_{19}.$$

In total, we obtain

$$I \leq |z + b|^2 + C_{20} (1 + |\Delta b|^2). \quad (37)$$

We estimate the third term of (35) as follows:

$$\begin{aligned} & \left| \sum_{k=1}^N \sum_{i=1}^n a_{ik} |z_{x_i} + b_{x_i}|^{p(x,t)-2} (z_{k,x_i} + b_{k,x_i}) b_{k,x_i} \right| \leq \\ & \leq \sum_{i=1}^n \sum_{k=1}^N |a_{ik}| \cdot |z_{x_i} + b_{x_i}|^{p(x,t)-2} |z_{k,x_i} + b_{k,x_i}| \cdot |b_{k,x_i}| \leq \\ & \leq \sum_{i=1}^n \sum_{k=1}^N |a_{ik}| \cdot |z_{x_i} + b_{x_i}|^{p(x,t)-1} \cdot |b_{x_i}| \leq C_{21} \sum_{i=1}^n |z_{x_i} + b_{x_i}|^{p(x,t)-1} \cdot |b_{x_i}| \leq \\ & \leq \varepsilon_2 \sum_{i=1}^n |z_{x_i} + b_{x_i}|^{p(x,t)} + C_{22}(\varepsilon_2) \sum_{i=1}^n |b_{x_i}|^{p(x,t)}. \end{aligned} \quad (38)$$

We estimate the fourth term of (35) as follows:

$$\begin{aligned} & \left| \sum_{k=1}^N g_k |z + b|^{q(x,t)-2} (z_k + b_k) b_k \right| \leq \sum_{k=1}^N |g_k| \cdot |z + b|^{q(x,t)-2} |z_k + b_k| \cdot |b_k| \leq \\ & \leq \sum_{k=1}^N |g_k| \cdot |z + b|^{q(x,t)-1} \cdot |b| \leq \varepsilon_3 |z + b|^{q(x,t)} + C_{23}(\varepsilon_3) |b|^{q(x,t)}. \end{aligned} \quad (39)$$

Using notation (17), from condition (Φ) , we get

$$\begin{aligned} J &:= \left| (\Phi(t)(z+b), b)_\Omega \right| = \left| \int_\Omega \sum_{k=1}^N \varphi_k \left((E(t)(z+b))_k \right) b_k \, dx \right| \leq \\ &\leq \int_\Omega \sum_{k=1}^N \varphi^0 \left| (E(t)(z+b))_k \right| \cdot |b_k| \, dx \leq C_{24} \int_\Omega |\mathbf{E}(z+b)| \cdot |b| \, dx. \end{aligned}$$

From the Cauchy-Bunyakowski-Schwarz's inequality, it follows

$$J \leq C_{25} \|\mathbf{E}(z+b); [L^2(\Omega)]^N\| \cdot \|b; [L^2(\Omega)]^N\|.$$

Then, from (22), we get $J \leq C_{26} \|z+b; [L^2(\Omega)]^N\| \cdot \|b; [L^2(\Omega)]^N\|$. Finally, from the Young inequality, we have that

$$J \leq C_{27} \left(\|z+b; [L^2(\Omega)]^N\|^2 + \|b; [L^2(\Omega)]^N\|^2 \right) = C_{27} \left(\int_\Omega |z+b|^2 \, dx + \int_\Omega |b|^2 \, dx \right). \quad (40)$$

Using (36)-(40), the second term (31) we estimate as

$$\begin{aligned} &\left| \left\langle S(t)(z+b), b \right\rangle \right| \leq \\ &\leq \int_\Omega \left[\varepsilon_1 |\Delta(z+b)|^2 + \varepsilon_2 \sum_{i=1}^n |z_{x_i} + b_{x_i}|^{p(x,t)} + \varepsilon_3 |z+b|^{q(x,t)} + C_{28} |z+b|^2 \right] dx + \\ &+ \int_\Omega \left[C_{29}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \left(|\Delta b|^2 + \sum_{i=1}^n |b_{x_i}|^{p(x,t)} + |b|^{q(x,t)} \right) + C_{30} |b|^2 \right] dx + C_{31}. \end{aligned} \quad (41)$$

Substituting the obtained estimates (34) and (41) into (31) and choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ sufficiently small, we obtain (29). \square

3 PROOF OF MAIN RESULTS

Let us prove Theorem 1 by the Faedo-Galerkin method.

Proof of Theorem 1. Step 1. Let $\{w^\mu\}_{\mu \in \mathbb{N}}$ be a set of all eigenfunctions of the problem

$$-\Delta w^j = \lambda_j w^j \quad \text{in } \Omega, \quad w^j|_{\partial\Omega} = 0, \quad j \in \mathbb{N}, \quad (42)$$

be also orthonormal in the space $L^2(\Omega)$,

$$\mathfrak{M}^N = \left\{ x \mapsto \left(\sum_{\mu=1}^m \alpha_{\mu 1}^m w^\mu(x), \dots, \sum_{\mu=1}^m \alpha_{\mu N}^m w^\mu(x) \right) \mid \alpha_{\mu k}^m \in \mathbb{R} \right\},$$

the number s is taken from condition (\mathbf{Z}) , and the spaces $\mathcal{W}_s, \mathcal{W}_s^*$ are taken from (11). From the choice of s and statement (25) it follows that \mathfrak{M}^N is a dense set everywhere in $[\mathcal{W}_s]^N$ and in $V(t)$, $t \in [0, T]$. Let $m \in \mathbb{N}$, $\tilde{u}^m := (\tilde{u}_1^m, \dots, \tilde{u}_N^m)$, where

$$\tilde{u}_k^m(x, t) := \sum_{\mu=1}^m \varphi_{\mu k}^m(t) w^\mu(x), \quad (x, t) \in Q_{0,T}, \quad k = \overline{1, N},$$

$\varphi^m := (\varphi_{11}^m, \varphi_{21}^m, \dots, \varphi_{m1}^m, \dots, \varphi_{1N}^m, \varphi_{2N}^m, \dots, \varphi_{mN}^m)$ is the solution to the Cauchy problem

$$\langle \tilde{u}_t^m(t), w^\mu \rangle + \langle S(t)(\tilde{u}^m(t) + b(t)), w^\mu \rangle = \langle F(t), w^\mu \rangle, \quad (43)$$

$$\varphi_{\mu k}^m(0) = \beta_{\mu k}^m, \quad k = \overline{1, N}, \quad \mu = \overline{1, m}. \quad (44)$$

Here $S(t)$ is taken from equality (18), $t \in [0, T]$, vector $\beta^m = (\beta_{11}^m, \beta_{21}^m, \dots, \beta_{m1}^m, \dots, \beta_{1N}^m, \beta_{2N}^m, \dots, \beta_{mN}^m) \in \mathbb{R}^{mN}$ selected so that the function $u_0^m := (u_{01}^m, \dots, u_{0N}^m)$, where $u_{0k}^m(x) := \sum_{\mu=1}^m \beta_{\mu k}^m w^\mu(x)$, $x \in \Omega$, $k = \overline{1, N}$, satisfied the condition $u_0^m \xrightarrow{m \rightarrow \infty} u_0$ strongly in H . It is clear that the following condition is satisfied.

$$\tilde{u}^m(0) = \left(\sum_{\mu=1}^m \varphi_{\mu 1}^m(0) w^\mu, \dots, \sum_{\mu=1}^m \varphi_{\mu N}^m(0) w^\mu \right) = u_0^m. \quad (45)$$

Let us denote $M := (M_{11}, M_{21}, \dots, M_{m1}, \dots, M_{1N}, M_{2N}, \dots, M_{mN})$, $M_{\mu k}(t) := \langle F_k(t), w^\mu \rangle$, $L := (L_{11}, L_{21}, \dots, L_{m1}, \dots, L_{1N}, L_{2N}, \dots, L_{mN})$,

$$L_{\mu k}(t, \varphi^m) := \langle (S(t)(\tilde{u}^m(t) + b(t)))_k, w^\mu \rangle, \quad k = \overline{1, N}, \quad \mu = \overline{1, m}, \quad t \in (0, T). \quad (46)$$

We will show that the conditions of the Caratheodory-LaSalle theorem (see Theorem 3.24 [4, p. 872]) are satisfied. From condition **(F)** we have that $M \in L^2(0, T; \mathbb{R}^{mN})$. The fact that the function L satisfies the L^∞ -condition of Caratheodory follows from the conditions of the theorem and statements of the type Propositions 1 and 2, applied separately to each term in (46). From the estimate (29) and the conditions $\alpha, a_0, g_0 > 0$ we obtain the estimate

$$(L(t, \varphi^m), \varphi^m)_{\mathbb{R}^{mN}} \geq -C_{32} \int_{\Omega} |\tilde{u}^m|^2 dx - C_{33} \beta(t) - C_{34} \geq -C_{35}(m) |\varphi^m|^2 - C_{33} \beta(t) - C_{34}, \quad (47)$$

where $C_{33}, C_{34}, C_{35} > 0$ do not depend on t, φ^m , and $\beta \in L^1(0, T)$. Then, the Caratheodory-LaSalle theorem implies that there exists a solution $\varphi^m \in H^1(0, T; \mathbb{R}^{mN})$ of problems (43), (44). Since $\partial\Omega \in C^{2s}$, then from (24) it follows that $\{w^\mu\}_{\mu \in \mathbb{N}} \subset \mathcal{W}_s \subset H^{2s}(\Omega)$. Therefore

$$\tilde{u}^m \in H^1(0, T; [H^{2s}(\Omega)]^N) \subset [H^1(Q_{0,T})]^N. \quad (48)$$

Step 2. Let us multiply the k -th coordinate of the μ -th equation from (43) by $\varphi_{\mu k}^m(t)$, sum up by $\mu = \overline{1, m}$ and $k = \overline{1, N}$, and integrate by $t \in (0, \tau) \subset (0, T)$. We obtain the equality

$$\int_{Q_{0,\tau}} (\tilde{u}_t^m, \tilde{u}^m)_{\mathbb{R}^N} dx dt + \int_0^\tau \left(L(t, \varphi^m(t)), \varphi^m(t) \right)_{\mathbb{R}^{mN}} dt = \int_{Q_{0,\tau}} (F, \tilde{u}^m)_{\mathbb{R}^N} dx dt, \quad \tau \in (0, T]. \quad (49)$$

Since (48) is hold, then $|\tilde{u}^m|^2 \in [W^{1,1}(Q_{0,T})]^N$ and $(|\tilde{u}^m|^2)_t = 2(\tilde{u}_t^m, \tilde{u}^m)$. Therefore, integrating by parts and considering (45), we obtain the equality

$$\int_{Q_{0,\tau}} (\tilde{u}_t^m, \tilde{u}^m) dx dt = \frac{1}{2} \int_{\Omega} |\tilde{u}^m(x, \tau)|^2 dx - \frac{1}{2} \int_{\Omega} |u_0^m(x)|^2 dx.$$

From the Young inequality, it follows that $|(F, \tilde{u}^m)| \leq |F| \cdot |\tilde{u}^m| \leq \frac{1}{2}|F|^2 + \frac{1}{2}|\tilde{u}^m|^2$. Therefore, using the estimate (29), from (49) we obtain the following:

$$\frac{1}{2} \int_{\Omega} |\tilde{u}^m(x, \tau)|^2 dx + \int_{Q_{0,\tau}} \left[\frac{\alpha}{4} |\Delta(\tilde{u}^m + b)|^2 + \frac{a_0}{2} \sum_{i=1}^n |\tilde{u}_{x_i}^m + b_{x_i}|^{p(x,t)} + \right.$$

$$\begin{aligned}
& + \frac{g_0}{2} |\tilde{u}^m + b|^{q(x,t)} dxdt \leq \frac{1}{2} \int_{\Omega} |u_0^m(x)|^2 dx + C_{36} + C_{37} \left\{ \int_0^{\tau} \beta(t) dt + \right. \\
& \left. + \int_{Q_{0,\tau}} [|F|^2 + |\tilde{u}^m + b|^2] dxdt \right\} \leq C_{36} + C_{38} \left\{ \mathbf{F}(\tau) + \int_{Q_{0,\tau}} |\tilde{u}^m|^2 dxdt \right\}, \quad (50)
\end{aligned}$$

where $\beta(t)$ is taken from (30), the constants $C_{36}, C_{38} > 0$ do not depend on m, τ ,

$$\mathbf{F}(\tau) := \int_{\Omega} |u_0(x)|^2 dx + \int_{Q_{0,\tau}} \left[|\Delta b|^2 + \sum_{i=1}^n |b_{x_i}|^{p(x,t)} + |b|^{q(x,t)} + |b|^2 \right] dxdt, \quad \tau \in (0, T]. \quad (51)$$

Let $y(t) = \int_{\Omega} |\tilde{u}^m(x, t)|^2 dx$, $t \in [0, T]$. Then from (50) we obtain inequality

$$y(\tau) \leq 2C_{36} + 2C_{38} \left\{ \mathbf{F}(\tau) + \int_0^{\tau} y(t) dt \right\}, \quad \tau \in (0, T].$$

We can show for this estimate that

$$\int_{\Omega} |\tilde{u}^m(x, \tau)|^2 dx \leq \left(2C_{36} + 2C_{38} \mathbf{F}(\tau) \right) e^{2C_{38}\tau}, \quad \tau \in (0, T]. \quad (52)$$

It is clear that from (52) follows the estimate

$$\int_{Q_{0,\tau}} |\tilde{u}^m|^2 dxdt \leq T \left(2C_{36} + 2C_{38} \mathbf{F}(\tau) \right) e^{2C_{38}T}, \quad \tau \in (0, T]. \quad (53)$$

Using (53), from (50) we obtain the estimates

$$\int_{Q_{0,\tau}} \left[|\Delta(\tilde{u}^m + b)|^2 + \sum_{i=1}^n |\tilde{u}_{x_i}^m + b_{x_i}|^{p(x,t)} + |\tilde{u}^m + b|^{q(x,t)} \right] dxdt \leq C_{39} + C_{40} \mathbf{F}(\tau), \quad (54)$$

$$\int_{Q_{0,\tau}} \left[|\Delta \tilde{u}^m|^2 + \sum_{i=1}^n |\tilde{u}_{x_i}^m|^{p(x,t)} + |\tilde{u}^m|^{q(x,t)} \right] dxdt \leq C_{41} + C_{42} \mathbf{F}(\tau), \quad \tau \in (0, T]. \quad (55)$$

Here the constants $C_{36}, \dots, C_{42} > 0$ do not depend on m, τ .

From (52)-(55) we get the existence of the subsequence $\{\tilde{u}^{m_j}\}_{j \in \mathbb{N}} \subset \{\tilde{u}^m\}_{m \in \mathbb{N}}$ such that

$$\tilde{u}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u} \quad * - \text{weakly in } L^\infty(0, T; H) \quad \text{and weakly in } U(Q_{0,T}). \quad (56)$$

Step 3. Let \mathbf{S} be taken from (19), $\langle \mathcal{F}, v \rangle_{U(Q_{0,T})} = \int_0^T \langle F(t), v(t) \rangle dt$, From estimates (22), (53), and (55) the following inequality follows $\|\mathbf{S}(\tilde{u}^m + b); [U(Q_{0,T})]^*\| \leq C_{43}$, from which it follows (possibly, when transitioning to a new sub-sequence) that:

$$\mathbf{S}(\tilde{u}^{m_j} + b) \xrightarrow{j \rightarrow \infty} \chi \quad \text{weakly in } [U(Q_{0,T})]^*. \quad (57)$$

Step 4. Let the numbers s, r^0 be taken from condition **(Z)**, the spaces $\mathcal{W}_s, \mathcal{W}_s^*$ are taken from equalities (11). Similarly as in [4], from (43), (55), and (27) we obtain

$$\|\tilde{u}_t^m; L^{\frac{r^0}{r^0-1}}(0, T; [\mathcal{W}_s^*]^N)\| \leq C_{44}, \quad (58)$$

where the constant $C_{44} > 0$ does not depend on m . Therefore (further, when moving on to a new sub-sequence, we will not indicate this separately)

$$\tilde{u}_t^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u}_t \text{ weakly in } L^{\frac{r^0}{r^0-1}}(0, T; [\mathcal{W}_s^*]^N). \quad (59)$$

Step 5. From (25) it follows that $V_- \overset{K}{\subset} H \bar{\circ} \mathcal{W}_s^*$. Also, from (26) and estimates (53), (55) we obtain the following

$$\|\tilde{u}^m; L^{r^0}(0, T; V_-)\| \leq C_{45} \|\tilde{u}^m; U(Q_{0,T})\| \leq C_{46}, \quad (60)$$

where the constant $C_{46} > 0$ does not depend on m .

From (60) and (58), the Aubin theorem (see Proposition 3.11 [4, p. 867]) and Lemma 1.18 [1, Part 2, §1.3] imply the following convergences:

$$\tilde{u}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u} \text{ in } L^2(0, T; H) \text{ and } C([0, T]; [\mathcal{W}_s^*]^N), \quad (61)$$

$$\tilde{u}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u} \text{ almost everywhere in } Q_{0,T}. \quad (62)$$

It is also clear that $V_- \overset{K}{\subset} [H_0^1(\Omega)]^N \bar{\circ} [\mathcal{W}_s^*]^N$. Therefore, from (60), (58), and the Aubin theorem follows the convergence $\tilde{u}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u}$ strongly in $L^2(0, T; [H_0^1(\Omega)]^N)$. Then, for each $i \in \{1, \dots, n\}$ we will have the following:

$$\int_{Q_{0,T}} |\tilde{u}_{x_i}^{m_j} - \tilde{u}_{x_i}|^2 dxdt \leq \|\tilde{u}^{m_j} - \tilde{u}; L^2(0, T; [H_0^1(\Omega)]^N)\|^2 \xrightarrow{j \rightarrow \infty} 0.$$

Therefore, $\tilde{u}_{x_i}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u}_{x_i}$ strongly in $[L^2(Q_{0,T})]^N$ and hence, from Lemma 1.18 [1, Part 2, §1.3] we obtain that

$$\tilde{u}_{x_i}^{m_j} \xrightarrow{j \rightarrow \infty} \tilde{u}_{x_i} \text{ almost everywhere in } Q_{0,T}, \quad i = \overline{1, n}. \quad (63)$$

Using (62) and (63) we prove the equality $\chi = S(\tilde{u} + b)$ (see (57)).

Step 6. Let's consider (43) for $m = m_j$, multiply this equality by an arbitrary function $\varphi \in C_0^1([0, T])$, integrate by $t \in (0, T)$ and integrate the first term by parts. In particular, by letting $j \rightarrow +\infty$, we obtain (20).

From (20) $\tilde{u}_t = \mathcal{F} - S\tilde{u} \in [U(Q_{0,T})]^*$. Therefore, $\tilde{u} \in W(Q_{0,T}) \subset C([0, T]; H)$. Thus, \tilde{u} is the solution to problem (1)-(3). \square

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У роботі досліджено мішану задачу для еволюційних систем четвертого порядку зі зсувом. Знайдено достатні умови існування слабкого розв'язку такої задачі.