

DOI: <https://doi.org/10.31861/bmj2025.02.06>

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CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM WITH ONE INVARIANT STRAIGHT LINE AND ONE INVARIANT CUBIC

In this paper the conditions for the existence of one invariant straight line and one invariant cubic in a cubic differential system with a singular point of a center or a focus type, when these curves have not intersecting points, are found. It is proved that the singular point is a center if and only if the first five Lyapunov quantities vanish. The center conditions were determined by using the method of Darboux integrability.

Key words and phrases: Cubic differential system, the center-focus problem, invariant straight line, invariant cubic, Lyapunov quantity.

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1 THE PROBLEM OF THE CENTER

We consider a planar system of differential equations

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1)$$

where $P(x, y)$ and $Q(x, y)$ are coprime polynomials with real coefficients in the real variables x, y . The degree n of this polynomial system is the maximum degrees of the polynomials P and Q , $n = \max\{\deg P(x, y), \deg Q(x, y)\}$. Associated to this polynomial differential system there is the polynomial vector field $\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$. If $n = 2$ ($n = 3$), then the system (1) is called a quadratic (respectively, a cubic) differential system.

Let $O(0, 0)$ be a singular point of differential system (1), i.e. $P(0, 0) = Q(0, 0) = 0$, and consider the linearization of (1) at $O(0, 0)$:

$$\frac{dx}{dt} = a_{10}x + a_{01}y, \quad \frac{dy}{dt} = b_{10}x + b_{01}y. \quad (2)$$

The most important question which is still open for planar systems of differential equations is the following one [6], [19]: under which conditions do the original system (1) and the linearized

УДК 517.925

2010 *Mathematics Subject Classification*: 34C05.

This work was supported by the National Agency for Research and Development of the Republic of Moldova under project number "25.80012.5007.76SE Qualitative and algebraic investigation of differential models".

system (2) have the same qualitative behavior and the same topological structure around a singular point $O(0,0)$?

This problem has been solved for (1) unless if the singular point $O(0,0)$ is of a center or a focus type, i.e. the eigenvalues of the linearized system are purely imaginary $\lambda_{1,2} = \pm i\beta$, $i^2 = -1$, $\beta \neq 0$ and $n \geq 3$. In this case, by using a nondegenerate transformation of variables and a time rescaling, the system (1) can be brought to the canonical form

$$\frac{dx}{dt} = y + \sum_{j=2}^n P_j(x, y) \equiv P(x, y), \quad \frac{dy}{dt} = -x + \sum_{j=2}^n Q_j(x, y) \equiv Q(x, y), \quad (3)$$

where P_j and Q_j are homogeneous polynomials of degree j with real variables and coefficients. A singular point $O(0,0)$ is either a focus or a center for (3), called *a weak focus*, *a fine focus* or *a monodromic singular point*.

The problem of distinguishing between a center or a focus is called *the problem of the center* or *the center-focus problem*. The interest in solving the problem of the center for differential systems (3) arose as part of investigation of the local 16th Hilbert problem.

The problem of the center was solved for some classes of cubic differential systems with invariant algebraic curves: four invariant straight lines [6], [17]; three invariant straight lines [6], [22]; two parallel invariant straight lines [13], [20]; two invariant straight lines and one invariant conic [3], [4], [5], [6]; two invariant straight lines and one invariant cubic [8], [9], [10], [15].

The goal of this paper is to obtain new center conditions for a cubic differential system with two invariant algebraic curves. The paper is organized as follows. In Section 2 we present some known results concerning the local integrability and the center variety. In Section 3 we discuss the relation between the existence of algebraic solutions and the Darboux integrability. In Section 4 we obtain necessary and sufficient conditions for a cubic system, with a singular point $O(0,0)$ of a center or a focus type, to have one invariant straight line and one irreducible invariant cubic such that these curves have not intersection points. In Section 5 we prove that the singular point $O(0,0)$ is a center if and only if the first five Lyapunov quantities vanish. Finally, we give four sets of necessary and sufficient conditions for a singular point $O(0,0)$ to be a center.

2 LYAPUNOV'S QUANTITIES

An approach to the problem of the center is to study the integrability of a differential system (3). Lyapunov [18] proved that the problem of the center is equivalent to the problem of local integrability of a differential system (3) in the neighborhood of a singular point $O(0,0)$. A singular point $O(0,0)$ is a center for (3) if and only if the system has in some neighborhood of $O(0,0)$ a nonconstant analytic first integral [1]

$$F(x, y) \equiv x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x, y) = C \quad (4)$$

or an analytic integrating factor of the form

$$\mu(x, y) = 1 + \sum_{k=1}^{\infty} \mu_k(x, y), \quad (5)$$

where F_k , μ_k are homogeneous polynomials of degree k .

According to [18] the problem of the center can be reduced to the problem of solving an infinite system of polynomial equations whose variables are parameters of the differential system. There exists a formal power series $F(x, y) = \sum F_k(x, y)$, defined in a neighbourhood of the origin, such that the rate of change of $F(x, y)$ along trajectories of system (3) is

$$\frac{dF}{dt} = L_1(x^2 + y^2)^2 + L_2(x^2 + y^2)^3 + \dots \quad (6)$$

where the quantities $L_k, k = 1, 2, \dots$ are polynomials in the coefficients of system (3) called *the Lyapunov (focus) quantities*. The stability of the origin is determined by the first nonvanishing Lyapunov quantity.

The origin is a fine focus of *order* m if $L_k = 0, k = \overline{1, m-1}$ and $L_m \neq 0$. In this case at most m small amplitude limit cycles can bifurcate from a fine focus of order m .

Theorem 1. *The origin is a center for differential system (3) if and only if all the Lyapunov quantities vanish ($L_k = 0, k = \overline{1, \infty}$).*

By the Hilbert's basis theorem, there is a natural number N such that $L_k = 0$ for all k if and only if $L_k = 0$ for all $k \leq N$. It is only necessary to find a finite number of Lyapunov quantities, though in any given case it is not known a priori how many are required. We come to the following Open Problem [6]:

Problem 1. *For any degree $n, n \geq 3$, of the differential system (3) to find such $N = N(n)$ that vanishing the first N Lyapunov quantities implies the existence of a center.*

The problem of the center was solved for: quadratic differential systems ($N = 3$); cubic symmetric differential systems ($N = 5$); the Kukles differential system ($N = 8$).

3 ALGEBRAIC SOLUTIONS AND DARBOUX INTEGRABILITY

An important problem concerning the integrability of differential systems (3) is whether the trajectories to (3) can be described implicitly by an algebraic formula, for example, $\Phi(x, y) = 0$, where Φ is a polynomial.

Definition 1. *An invariant algebraic curve of system (3) is the solution set in \mathbb{C}^2 of an equation $\Phi(x, y) = 0$, where Φ is a polynomial in x, y with complex coefficients such that*

$$\mathcal{X}\Phi = P(x, y)\frac{\partial\Phi}{\partial x} + Q(x, y)\frac{\partial\Phi}{\partial y} = K\Phi, \quad (7)$$

for some polynomial in $x, y, K = K(x, y)$ of degree $n - 1$ with complex coefficients, called the cofactor of the invariant algebraic curve $\Phi(x, y) = 0$.

Definition 2. *An invariant algebraic curve $\Phi(x, y) = 0$ is called an algebraic solution of (3) if $\Phi(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$.*

Let $\Phi = 0$ be an algebraic solution of degree m for system (3). Then this algebraic solution is:

- 1) an *invariant straight line* if $m = 1$, and it has the form $a_{10}x + a_{01}y + 1 = 0$;
- 2) an *invariant conic* if $m = 2$, and it has the form

$$a_{20}x^2 + a_{11}xy + c_{02}y^2 + a_{10}x + a_{01}y + 1 = 0;$$

3) an *invariant cubic* if $m = 3$, and it has the form

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0.$$

Since Darboux found in 1878 connections between invariant algebraic curves and the existence of first integrals of planar polynomial differential systems, the theory of invariant algebraic curves is still full of open questions [21], [23].

Problem 2. Give a method to find an upper bound M to the degree of the algebraic solutions for a fixed polynomial system (1) of degree $n \geq 2$.

Problem 3. What is the maximum number $\alpha(n)$ of algebraic invariant curves in the set of all polynomial differential systems of degree $n > 1$ having finitely many invariant algebraic curves?

For a given polynomial system (1) of degree n the calculation of the invariant algebraic curves is a very hard problem because in general we don't have any evidence about the number of invariant algebraic curves and the degree of a curve.

Invariant algebraic curves are central object in the theory of integrability of polynomial differential systems. This motivates the growing interest of researchers in application and development of the algebraic method of integrability for polynomial systems. This kind of integrability is usually called *Darboux integrability*, and it provides a link between the *integrability of polynomial differential systems and the number of invariant algebraic curves* they have [6], [21].

Suppose the polynomial differential system (1) of degree n has q invariant algebraic curves $\Phi_j(x, y) = 0$, $j = 1, \dots, q$. Darboux proposed to search for a first integral (an integrating factor) in the form

$$\Phi_1^{\alpha_1} \Phi_2^{\alpha_2} \dots \Phi_q^{\alpha_q}, \quad (8)$$

where Φ_j are invariant algebraic curves and $\Phi_j \in C[x, y]$, $\alpha_j \in C$.

Problem 4. For polynomial systems of degree n determine the relations between the number q of invariant algebraic curves, their degrees and the existence of first integrals.

A partial answer to this problem was given by Darboux in the following Theorem [21].

Theorem 2. Suppose system (1) has q distinct irreducible invariant algebraic curves $\Phi_j = 0$, $j = 1, \dots, q$. If $q \geq \frac{1}{2}n(n+1)$, then either we have a Darboux first integral or a Darboux integrating factor.

The method of Darboux is very useful and elegant one to prove integrability for some families of differential systems if we have "enough" invariant algebraic curves. This motivates the following problem for polynomial differential systems (3).

Problem 5. Find the polynomial differential systems (3) with a fewer number of invariant algebraic curves than $n(n+1)/2$ for which a singular point $O(0, 0)$ is a center.

In [21] Schlomiuk proved that the method of Darboux can be applied to prove centers in all cases of the quadratic systems: any quadratic system ($n = 2$) with a singular point of a center type is Darboux integrable. The Darboux integrability conditions for some families of cubic systems having invariant algebraic curves were found in [2], [11], [12], [14], [16].

4 A CUBIC SYSTEM WITH TWO ALGEBRAIC SOLUTIONS

We consider the cubic system of differential equations $\{(3), (n = 3)\}$ with a singular point $O(0, 0)$ of a center or a focus type, written in the form

$$\begin{aligned} \frac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x, y), \\ \frac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y), \end{aligned} \quad (9)$$

in which variables and coefficients are assumed to be real. For system (9) we study the following problems:

(i) Find the subclass of systems (9) which has an invariant straight line $l_1 = 0$ and an invariant cubic $\Phi = 0$, without intersections points, i.e. $l_1 \cap \Phi = \emptyset$.

(ii) For this subclass find the integrability conditions such that the singular point $O(0, 0)$ is a center.

In this section we give necessary and sufficient conditions for the existence of one invariant straight line and one invariant cubic in a cubic differential system (9), such that $l_1 \cap \Phi = \emptyset$. Suppose system (9) has a real invariant straight line $l_1 \equiv a_1x + b_1y + 1 = 0$, $(a_1, b_1) \neq 0$. Then by rotating the system of coordinates $(x \rightarrow x \cos \varphi - y \sin \varphi, y \rightarrow x \sin \varphi + y \cos \varphi)$ and rescaling the axes of coordinates $(x \rightarrow \alpha x, y \rightarrow \alpha y)$, we can make the line to be $1 - x = 0$.

In [7] it was proved the following Lemma

Lemma 1. *The cubic differential system (9) has an invariant straight line $1 - x = 0$ if and only if the following set of conditions holds*

$$k = -a, \quad m = -c - 1, \quad p = -f, \quad r = 0. \quad (10)$$

When conditions (10) hold, the cubic system (9) can be written as follows

$$\begin{aligned} \frac{dx}{dt} &= (1 - x)(y + xy + ax^2 + cxy + fy^2) \equiv P(x, y), \\ \frac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y). \end{aligned} \quad (11)$$

We shall find the conditions on the coefficients of cubic system (11) under which the system has one real irreducible invariant cubic of the form

$$\Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0, \quad (12)$$

where $a_{30}, a_{21}, a_{12}, a_{03} \in \mathbb{R}$ and $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$, such that these curves have not intersection points, i.e. $\Phi(1, y) \neq 0$.

The equation

$$\Phi(1, y) \equiv a_{03}y^3 + (a_{12} + 1)y^2 + a_{21}y + a_{30} + 1 = 0 \quad (13)$$

has not real solutions in the following cases:

- (A) $a_{03} = 0, a_{12} = -1, a_{21} = 0, a_{30} \neq -1$;
- (B) $a_{03} = 0, a_{12} \neq -1, a_{21}^2 - 4(a_{30} + 1)(a_{12} + 1) < 0$.

Theorem 3. *The cubic differential system (11) has an invariant straight line $1 - x = 0$ and an invariant cubic (12) without intersection points if and only if one the following sets of conditions (c_1) – (c_7) holds:*

- (c₁) $f = a + d$, $g = (2b + 2c + 3)/2$, $k = -a$, $l = (-a - d)/2$, $m = -c - 1$, $n = (-2b - c - 2)/2$,
 $p = (-a - d)$, $q = a/2$, $r = s = 0$;
- (c₂) $a = k = r = 0$, $d = -2l$, $f = -2l$, $m = -c - 1$, $p = 2l$, $q = l(2b + 2c - 2g + 3)$,
 $n = (-2b - c - 2)/2$, $s = (8g - 4b^2 - 10bc + 4bg - 14b - 6c^2 + 6cg - 17c - 12)/6$;
- (c₃) $a = k = r = 0$, $d = f$, $l = (3f)/2$, $m = -c - 1$, $n = [3(c + 1)]/2$, $p = -f$, $q =$
 $[f(2g - 2b - 2c + 1)]/2$, $s = (4bg - 4b^2 - 10bc - 6c^2 + 6cg + 3c + 3)/6$;
- (c₄) $c = (fa_{30} - f - a)/(a + f)$, $d = 2(a + 2f)$, $g = (3aa_{30} + 2ab - 2a + 5fa_{30} + 2bf - 2f)/[2(a +$
 $f)]$, $k = -a$, $l = 2f$, $m = (-fa_{30})/(a + f)$, $n = [f(8a^2 + 16af + 3a_{30} + 8f^2)]/[2(a + f)]$,
 $p = -f$, $q = 2ab - 2a + 5fa_{30} + 2bf - 2f$, $r = 0$, $s = [a_{30}(2ab - 3a + 3fa_{30} + 2bf - 2f)]/[2(a + f)]$;
- (c₅) $a = (-2l)/(c + 2)$, $d = -a$, $f = p = r = s = 0$, $g = (2b - c - 3)/2$, $k = -a$, $m = -c - 1$,
 $n = (2bc + 2b - c^2 - 3c - 2)/2$, $q = (-l)/(c + 2)$;
- (c₆) $d = 2a$, $f = l = p = r = 0$, $k = -a$, $m = -c - 1$, $n = [(2b - c - 2)(c + 1)]/2$, $q =$
 $a(2b - c - 3)$, $s = [(2b - c - 2g - 3)(c - 2b + 4)]/6$;
- (c₇) $p = r = 0$, $a = (4a_{12}^2 - 4ca_{12} - 4a_{12} + a_{21}^2)/(2a_{21})$, $d = (a_{21}^2 - 2a_{12}^2 + 2ca_{12} + 2a_{12})/a_{21}$,
 $g = (3a_{21}^2 - 12a_{12}^2 + 8ba_{12} + 8ca_{12})/(8a_{12})$, $k = -a$, $l = [a_{12}(1 - a_{12}^2 + ca_{12} + c)]/a_{21}$,
 $m = -c - 1$, $n = (3 - 6a_{12}^2 + 2ba_{12} + 5ca_{12} + a_{12} + 3c)/2$, $q = (4a_{12}^2 - 9a_{12}a_{21}^2 - 4ca_{12} - 4a_{12} +$
 $4ba_{21}^2 + 7ca_{21}^2 + 3a_{21}^2)/(4a_{21})$, $s = (4a_{12} - 4a_{12}^2 - 4a_{12}a_{21}^2 + 4ca_{12} + 2ba_{21}^2 + 3ca_{21}^2)/(8a_{12})$.

Proof. We consider the cubic curve (12). By Definition 1, the curve (12) is invariant for (11) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01} \in \mathbb{R}$ such that

$$P(x, y) \frac{\partial \Phi}{\partial x} + Q(x, y) \frac{\partial \Phi}{\partial y} \equiv \Phi(x, y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y). \quad (14)$$

Identifying the coefficients of the monomials $x^i y^j$ in (14), we reduce this identity to a system of fifteen equations

$$\{U_{ij} = 0, \quad i + j = 3, 4, 5\} \quad (15)$$

for the unknowns $a_{30}, a_{21}, a_{12}, a_{03}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ and the coefficient of system (11).

To prove the theorem we solve the algebraic system of equations (15) in Cases (A) and (B).

1) In Case (A), we suppose that

$$a_{03} = 0, \quad a_{12} = -1, \quad a_{21} = 0, \quad a_{30} \neq -1$$

and the cubic curve (12) has the form $\Phi(x, y) \equiv x^2 + y^2 - xy^2 + a_{30}x^3 = 0$.

When $i + j = 3$, from (15), we find that $c_{01} = -2b - 1$, $c_{10} = 2a$, $a_{30} = (2g - 2b - 2c - 3)/3$, $f = a + d$, and when $i + j = 4$, we obtain that $c_{20} = [a(2g - 2b - 2c - 9)]/3$, $c_{11} = -c - 2n - 1$, $c_{02} = -a - d - 2l$, $s = (3n - 2b^2 - 5bc + 2bg - 4b - 3c^2 + 3cg - 7c + 4g - 3)/3$, $q = (3l - 2ab - 2ac + 2ag - 3bd - 3cd + 3dg - 3d)/3$.

In this case, the equation $U_{50} = 0$ of (15) looks as $U_{50} \equiv af_1 f_2 = 0$, where

$$f_1 = b + c - g, \quad f_2 = 2b + 2c - 2g + 3.$$

Assume that $f_1 = 0$, then $g = b + c$. In this case the cubic curve (14) is reducible.

Assume that $f_1 \neq 0$ and let $f_2 = 0$. Then the equations of (15) yield

$$l = (-a - d)/2, \quad n = (-c - 2 - 2b)/2, \quad g = (2b + 2c + 3)/2.$$

In this case we obtain the set of conditions (c₁) and the invariant cubic is

$$x^2 + y^2 - xy^2 = 0.$$

Assume that $f_1 f_2 \neq 0$ and let $a = 0$. Then the equations of (15) yield

$$n = (-c - 2 - 2b)/2, \quad d = -2l.$$

In this case we obtain the set of conditions (c_2) and the invariant cubic looks

$$3x^2 + 3y^2 - 3xy^2 + (2g - 2b - 2c - 3)x^3 = 0, \quad g - b - c \neq 0.$$

2) In Case (B), we suppose that

$$a_{03} = 0, \quad a_{12} \neq -1, \quad \Delta = a_{21}^2 - 4(a_{30} + 1)(a_{12} + 1) < 0$$

and the cubic curve (12) has the form $\Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 = 0$.

When $i + j = 3$, from (15), we find that $c_{01} = a_{12} - 2b$, $c_{10} = 2a - a_{21}$, $d = (3a_{21} - 2a + 2f)/2$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$, and when $i + j = 4$, we obtain that $c_{02} = fa_{12} - 2l$, $c_{11} = (c + 2 - a_{12})a_{12} - 2n$, $c_{20} = aa_{12} - 3a_{12}a_{21} - 3fa_{12} + ba_{21} + 2ca_{21} + 3fa_{30} - 2f + 2l - 2q$, $s = (2aa_{21} + 8a_{12}^2 - 8a_{12}a_{30} - 4ba_{12} - 6ca_{12} - a_{21}^2 - 6fa_{21} + 4ba_{30} + 6ca_{30} - 4c + 4n - 4)/4$, $q = (2aa_{12} - 2aa_{30} + 4a - 9a_{12}a_{21} - 6fa_{12} + a_{21}a_{30} + 4ba_{21} + 6ca_{21} + 6fa_{30} - 4f + 4l)/4$.

In this case, the equation $U_{05} = 0$ of (15) looks $U_{05} \equiv fa_{12}(a_{12} + 1) = 0$, where $a_{12} + 1 \neq 0$.

Assume that $a_{12} = 0$, then $F_{23} \equiv (l - 2f)a_{21} = 0$. If $a_{21} = 0$, then $a_{30}(a_{30} + 1) \neq 0$ and $a = 0$, $n = (3(c + 1))/2$, $l = (3f)/2$. We obtain the set of conditions (c_3) for the existence of the invariant cubic

$$3x^2 + 3y^2 + 2(g - b - c)x^3 = 0, \quad 2(g - b - c) + 3 > 0.$$

Assume that $a_{21} \neq 0$ and let $l = 2f$. In this case we express n from the equation $U_{32} = 0$ and calculate the resultant of the polynomials U_{50} and U_{41} with respect to c . We obtain that

$$Res(U_{50}, U_{41}, c) = 2a_{21}g_1g_2\Delta,$$

where $g_1 = a_{21} - 2f - 2a$, $g_2 = a_{21}^2 + a_{30}^2 \neq 0$, $\Delta = a_{21}^2 - 4a_{30} - 4$.

If $a_{21} = 2(a + f)$, then $g_1 = 0$. We get the set of conditions (c_4) and the invariant cubic

$$x^2 + y^2 + a_{30}x^3 + 2(a + f)x^2y = 0, \quad (a + f)^2 - a_{30} - 1 < 0.$$

Assume now that $a_{12} \neq 0$ and let $f = 0$. If $a_{21} = 0$, then $a \neq 0$ and the equations of (15) yields $a_{12} = c + 1$, $a_{30} = 0$, $a = (-2l)/(c + 2)$, $n = [(2b - c - 2)(c + 1)]/2$. In this case we find the set of conditions (c_5) for the existence of the invariant cubic

$$x^2 + y^2 + (c + 1)xy^2 = 0, \quad c + 2 > 0.$$

Suppose that $a_{21} \neq 0$. In this case we express l from $U_{23} = 0$, n from $U_{32} = 0$ and calculate the resultant of the polynomials U_{50} and U_{41} with respect to c . We obtain that

$$Res(U_{50}, U_{41}, c) = 4a_{21}h_1h_2h_3\Delta,$$

where $h_1 = a_{12} - c - 1$, $h_2 = 4a_{12}a_{30} - a_{21}^2$, $h_3 = (a_{12} - a_{30})^2 + a_{21}^2 \neq 0$, $\Delta = a_{21}^2 - 4a_{12}a_{30} - 4a_{12} - 4a_{30} - 4$.

If $h_1 = 0$, then $a_{12} = c + 1$ and the equations $U_{50} = 0$, $U_{41} = 0$ yield $a_{21} = 2a$. We obtain the set of conditions (c_6) for the existence of the invariant cubic

$$3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2 = 0,$$

where $3a^2 + 2bc + 4b - c^2 - 2cg - 8c - 4g - 12 < 0$.

If $h_1 \neq 0$ and $h_2 = 0$, then $a_{30} = a_{21}^2/(4a_{12})$ and the equations $U_{50} = 0$, $U_{41} = 0$ of (15) yield $a = (4a_{12}^2 - 4ca_{12} - 4a_{12} + a_{21}^2)/(2a_{21})$. We obtain the set of conditions (c_7) for the existence of the invariant cubic

$$4a_{12}(x^2 + y^2) + a_{21}^2x^3 + 4a_{12}a_{21}x^2y + 4a_{12}^2xy^2 = 0, \quad a_{12}(4a_{12}^2 + 4a_{12} + a_{21}^2) > 0.$$

□

5 THE CENTER-FOCUS PROBLEM FOR A CUBIC SYSTEM WITH TWO ALGEBRAIC SOLUTIONS

In this section we solve the center-focus problem for a cubic differential system (11) with one invariant straight line and one invariant cubic, in the case when these invariant algebraic curves have not real intersection points.

Lemma 2. *The following sets of conditions are sufficient conditions for the origin to be a center of system (11):*

$$(i) \quad c = 1 - 2b, \quad d = 2(a + 2f), \quad g = (3a - 3ab - 4bf + 4f)/f, \quad k = -a, \quad l = 2f, \quad m = 2(b - 1), \\ n = 4af - 3b + 4f^2 + 3, \quad p = -f, \quad q = 8(a - ab - bf + f), \quad r = 0, \quad s = (4ab^2 - 7ab + 3a + 4b^2f - 8bf + 4f)/f;$$

$$(ii) \quad c = 2(b - 1), \quad d = -a, \quad f = n = p = r = s = 0, \quad l = -ab, \quad g = (-1)/2, \quad k = -a, \\ m = 1 - 2b, \quad q = a/2;$$

$$(iii) \quad d = 2a, \quad f = l = p = r = 0, \quad k = -a, \quad m = -c - 1, \quad n = [(2b - c - 2)(c + 1)]/2, \quad q = \\ a(2b - c - 3), \quad s = [(2b - c - 2g - 3)(c - 2b + 4)]/6;$$

$$(iv) \quad a = [2(2b + 2c + 1)(2b - c + 2g - 1)]/(9a_{21}), \quad d = [2(2b + 2c + 1)(c - 2b + 4g + 4)]/(9a_{21}), \\ k = -a, \quad f = p = r = 0, \quad l = [2(2b + 2c + 1)(c - 2b + 2)(b + c + 2)]/(27a_{21}), \quad m = -c - 1, \\ n = [(c - 2b + 2)(b + c + 2)]/3, \quad q = [2(cg - 2bg - 1)(2b + 2c + 1)]/(9a_{21}), \quad s = [(c - 2b)(g + 2) + 4g - 1]/9, \\ 9a_{21}^2 - 4(2b + 2c + 1)(2g + 1) = 0.$$

Proof. In the case (i), the cubic system (11) has the algebraic curves

$$1 - x = 0, \quad f(x^2 + y^2) + 2f(a + f)x^2y + 2(a - ab + f - bf)x^3 = 0$$

and the integrating factor

$$\mu = (1 - x)(f(x^2 + y^2) + 2f(a + f)x^2y + 2(a - ab + f - bf)x^3)^2.$$

In the case (ii), the cubic system (11) has the algebraic curves

$$1 - x = 0, \quad x^2 + y^2 + 2bxy^2 - xy^2 = 0$$

and the integrating factor

$$\mu = \frac{1}{(1 - x)^{1/2}(x^2 + y^2 + 2bxy^2 - xy^2)^{3/2}}.$$

In the case (iii), the cubic system (11) has the algebraic curves

$$1 - x = 0, \quad 3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2 = 0$$

and the first integral

$$(1 - x)^{c-2b+1}(3x^2 + 3y^2 + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2) = C.$$

In the case (iv), the cubic system (11) has the algebraic curves

$$1 - x = 0, \quad 12(2b + 2c + 1)(x^2 + y^2) + 9a_{21}^2x^3 + 12(2b + 2c + 1)a_{21}x^2y + 4(2b + 2c + 1)^2xy^2 = 0$$

and the integrating factor

$$\mu = \frac{(1 - x)^{-1/2}}{(12(2b + 2c + 1)(x^2 + y^2) + 9a_{21}^2x^3 + 12(2b + 2c + 1)a_{21}x^2y + 4(2b + 2c + 1)^2xy^2)^{3/2}}.$$

□

Theorem 4. *Let the cubic differential system (11) have one invariant straight line $1 - x = 0$ and one invariant cubic (12) without intersection points. Then the singular point $O(0, 0)$ is a center if and only if the first five Lyapunov quantities vanish.*

Proof. To prove the theorem, we compute the first five Lyapunov quantities L_j , $j = 1, \dots, 5$ in each set of conditions (c_1) – (c_7) obtained in Section 4 by using the algorithm described in [6]. In the expressions for L_j we will neglect the denominators and non-zero factors.

In Case (c_1) the first Lyapunov quantity looks $L_1 = 2ac + 3a - 2bd - d$. Assume that $a \neq 0$, then $L_1 = 0$ yields $c = (d - 3a + 2bd + d)/(2a)$. The second Lyapunov quantity is $L_2 = 32a^4 + 48a^3d - 32a^2b^2 - 12ba^2 + 16a^2d^2 - a^2 - 32ab^2d - 22abd - 3ad - 8b^2d^2 - 8bd^2 - 2d^2$.

Suppose that $L_2 = 0$. This equation admits the following parametrization

$$a = (128u^2 - v^2 + 1)/(128u), \quad d = (v^2 - 64u^2 - 1)/(64u),$$

$$b = (128u^2v + 128u^2 - v^3 - 5v^2 + v + 5)/(1024u^2).$$

The third Lyapunov quantity is $L_3 = e_1e_2$, where $e_1 = v + 1$, $e_2 = 1684(3v - 1)(v - 5)u^4 + 192(3v^2 + 16v - 3)(v^2 - 1)u^2 - 3(v + 5)(v + 1)^3(v - 1)^2$, $e_3 = 128u^2 - v^2 + 1 \neq 0$.

If $e_1 = 0$, then $v = -1$. In this case the cubic system (11) has two invariant straight lines $1 - x = 0$, $1 - ay = 0$ and one invariant cubic $x^2 + y^2 - xy^2 = 0$. For this case the center-focus problem was solved in [15].

Assume that $e_1 \neq 0$ and let $e_2 = 0$. We calculate the Lyapunov quantities L_4 , L_5 and the resultant of the polynomials e_2 and L_4 with respect to u . We obtain that

$$Res(L_4, e_2, u) = (v - 5)^2(v + 5)^6(3v - 1)^2(v - 1)^{16}(5v - 11)^2e_1^{22}j_1^2,$$

where $j_1 = 6615v^6 + 20055v^5 - 14584v^4 + 487314v^3 - 672085v^2 + 42135v + 188150$.

If $v = 5$, then $e_2 = 0$ yields $u^2 = 45/304$ and $L_4 \neq 0$. If $v = -5$, then $e_2 = 0$ implies $u^2 = 9/640$ and $L_4 \neq 0$. If $v = 1/3$ or $v = 1$, then $e_2 = 0$ has not real solutions.

If $v = 11/5$, then $e_2 = r_1r_2r_3$, where $r_1 = 1225u^2 - 54$, $r_2 = 10u - 3$, $r_3 = 10u + 3$.

When $r_1 = 0$ we obtain that $L_4 \neq 0$. If $r_2 = 0$, then the cubic system (11) has two invariant straight lines $1 - x = 0$, $2 - y = 0$ and one invariant cubic $x^2 + y^2 - xy^2 = 0$. If $r_3 = 0$, then (11) has two invariant straight lines $1 - x = 0$, $2 + y = 0$. The center-focus problem was solved in [15].

Assume that $j_1 = 0$. The equations $j_1 = 0$ and $e_2 = 0$ have real solutions, but $L_5 \neq 0$. In this case the origin is a focus.

In Case (c_2) the first Lyapunov quantity looks $L_1 = l(2b + 1)$. If $l = 0$, then the cubic system (11) has two parallel invariant straight lines $1 - x = 0$, $1 + (1 + c)x = 0$ and one invariant cubic $3(x^2 + y^2) - 3xy^2 + (2g - 2b - 2c - 3)x^3 = 0$. For this case the center-focus problem was solved in [8]. If $l \neq 0$ and $b = (-1)/2$, then $L_1 = L_2 = \dots L_5 = 0$. In this case the right hand sides of (11) have a common factor $1 + (c + 1)x - 2ly$, in contradiction with assumption that $P(x, y)$ and $Q(x, y)$ are coprime polynomials.

In Case (c_3) the first Lyapunov quantity looks $L_1 = f(2b - 3)$. If $f = 0$, then the cubic (11) has two parallel invariant straight lines $1 - x = 0$, $1 + (1 + c)x = 0$ and one invariant cubic. The center-focus problem was solved in [8]. If $f \neq 0$ and $b = 3/2$, then $L_1 = L_2 = 0$. In this case the right hand sides of (11) have a common factor $1 + (c + 1)x + fy$, in contradiction with assumption that $P(x, y)$ and $Q(x, y)$ are coprime polynomials.

In Case (c_4) the first Lyapunov quantity looks $L_1 = f(2f - 2ab + 2a - fa_{30} - 2bf)$. If $f = 0$, then we are in conditions (iii) of Lemma 2 ($c = -1$). If $f \neq 0$ and $a_{30} = [2(a - ab - bf + f)]/f$, then $L_1 = L_2 = \dots L_5 = 0$. In this case we have the set of conditions (i) from Lemma 2.

In Case (c_5) the first Lyapunov quantity looks $L_1 = 4l(c - 2b + 2)$. If $l = 0$, then the cubic system (11) has two parallel invariant straight lines $x = 1$, and one invariant cubic. If $l \neq 0$ and

$c = 2(b - 1)$, then have the conditions (ii) from Lemma 2.

In Case (c_6) we obtain that $L_1 = L_2 = \cdots = L_5 = 0$. In this case we get the set of conditions (iii) from Lemma 2.

In Case (c_7) the first Lyapunov quantity looks $L_1 = (3a_{12} - 2b - 2c - 1)(a_{12} - c - 1)$. If $a_{12} = c + 1$, then we have the set of conditions (iii) from Lemma 2. Assume that $a_{12} \neq c + 1$ and let $a_{12} = (2b + 2c + 1)/3$. We are in conditions (iv) of Lemma 2. \square

Taking into account Lemma 2 and Theorem 4, and excluding the cases when the cubic system (11) has two invariant straight lines and one invariant cubic, we give the necessary and sufficient conditions for the origin to be a center in the following Theorem.

Theorem 5. *The singular point $O(0,0)$ is a center for a cubic differential system (11), with one invariant straight line and one invariant cubic without intersection points, if and only if one of the sets of conditions (i)-(iv) holds.*

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Received 26.11.2025

Козма Д. *Center conditions for a cubic differential system with one invariant straight line and one invariant cubic* // Буковинський матем. журнал — 2025. — Т.13, №2. — С. 47–57.

In this paper the conditions for the existence of one invariant straight line and one invariant cubic in a cubic differential system with a singular point of a center or a focus type, when these curves have not intersecting points, are found. It is proved that the singular point is a center if and only if the first five Lyapunov quantities vanish. The center conditions were determined by using the method of Darboux integrability.