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ISOTROPIC PROBLEM WITH NON HOMOGENEOUS DIRICHLET BOUNDARY CONDITION AND L^1-DATA

The purpose of this paper is to study an isotropic problem with non homogeneous Dirichlet boundary condition and L^1 data in a variable-exponent Sobolev space. We start by showing the existence and uniqueness of the weak solution when the source term is bounded. Finally, we use a problem-based approach to prove the existence and uniqueness of entropy solution when the source term is integrable.

Key words and phrases: Sobolev spaces, variable exponent, weak solution, entropy solution.

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1 Introduction

The study of problems solving partial differential equations has undergone a great revolution in the field of science, particularly those using Sobolev spares with variable exponents. These are used to model several physical phenomena. Indeed, in chemistry, they are used to model fluids that change their chemical properties under the influence of an electric field [12, 5, 9, 13], in biology, Leray-Lions type operators with $p(.)$ growth also appear. They are also used in image restoration [4].
The aim of this paper is to study the existence and uniqueness of weak and/or entropy solution under several conditions on the data for the following isotropic problem with non homogeneous Dirichlet boundary condition:

$$\begin{cases} -div(a(x,\nabla u)) + |u|^{p(x)-2}u = f \text{ in } \Omega \\ u = constant \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N > 1)$, $\partial\Omega$ represents the boundary of Ω and $a(.,.) : \bar{\Omega} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function.
In the literature, there are many works analogous to problem (1). For example in [3] the authors studied the existence and uniqueness of weak solution of a perturbed problem on its right side,

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which depends on x and u. In [11] the authors studied the existence and uniqueness of the weak solution when $f \in L^{\infty}(\Omega)$ and the entropy solution when $f \in L^1(\Omega)$ of the following problem:

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = f \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

The authors in [10] solved the following problem:

$$\begin{cases} b(u) - \operatorname{div}(a(x,\nabla u)) = f \quad \text{in } \Omega \\ a(x,\nabla u).\eta \ = -|u|^{p(x)-2}u \ \text{ on } \partial\Omega, \end{cases}$$

where η denote the outer unit normal vector on $\partial\Omega$ by getting the uniqueness and existence of a weak solution when $f \in L^{\infty}(\Omega)$ and of an entropy solution when $f \in L^1(\Omega)$. Based on results already found for these authors, we study the problem (1). The novelty of this work is on the boundary condition and the L^1- condition on the datum f. Indeed, Boureanu (see [2, 3]) studied the kind of problem that we study but with f which depends on x and u where they proved the existence and multiplicity of weak solutions. In this work, we are interested with the existence and uniqueness of entropy solution. The remaining part of this paper is organized as follows: In Section 2, we introduce some useful preliminary results. In Section 3, we present the necessary assumptions on the data in order to address problem (1). In Section 4, we prove the existence and uniqueness of weak solution for $f \in L^{\infty}(\Omega)$. Finally, in Section 5 we prove the existence and uniqueness of entropy solution for $f \in L^1(\Omega)$.

2 Preliminary results

We denote $C_+(\bar\Omega) = \left\{ p(x) \in C(\bar\Omega;\mathbb{R}) : \inf_{x\in\bar\Omega} p(x) > 1 \right\}$.
$\forall p \in C_+(\bar\Omega)$, we denote

$$p^+ = \sup_{x\in\bar\Omega} p(x) \text{ and } p^- = \inf_{x\in\bar\Omega} p(x).$$

Let $p(.) \in C_+(\bar\Omega)$. We set the variable exponent Lebesgue space denoted by $L^{p(\cdot)}(\Omega)$ as follow:

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \text{ mesurable with real values such that } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}.$$

This space is equipped with the following Luxemburg norm:

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1 \right\}.$$

The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)}\right)$ is a reflexive and separable Banach space if $p \in C_+(\bar\Omega)$ and $p^+ < \infty$.
We also define the convex modular as follows:

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}dx,$$

with $u : \Omega \longrightarrow \mathbb{R}$ a measurable function.

Proposition 1. *[8] If $\Omega \subset \mathbb{R}^N$ is a bounded open subset and p_1, $p_2 \in C_+(\bar{\Omega})$ such as $p_1 \leq p_2$ in Ω, then the following injection $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ is continuous.*

We denote by $L^{p'(.)}(\Omega)$ the conjugate space of $L^{p(.)}(\Omega)$ with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, $\forall\ x \in \Omega$.

Theorem 1. *[8] $\forall u \in L^{p(.)}(\Omega)$ and $\forall v \in L^{p'(.)}(\Omega)$, the following Hölder type inequality holds:*

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{L^{p(.)}(\Omega)} \|v\|_{L^{p'(.)}(\Omega)}.$$

In particular, $\left|\int_\Omega uvdx\right| \leq 2\|u\|_{L^{p(.)}(\Omega)} \|v\|_{L^{p'(.)}(\Omega)}$.

Let's set the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\}.$$

We also define the associated norm and the convex modular to this space as follows:
If $u \in W^{1,p(.)}(\Omega)$,

$$\|u\|_{1,p(.)} = \|u\|_{L^{p(.)}(\Omega)} + \|\nabla u\|_{L^{p(.)}(\Omega)},$$

and

$$\rho_{1,p(.)}(u) = \int_\Omega \left[|\nabla u|^{p(x)} + |u|^{p(x)}\right] dx.$$

Proposition 2. *[8] The space $\left(W^{1,p(.)}(\Omega); \|.\|_{W^{1,p(.)}(\Omega)}\right)$ is a reflexive and separable Banach space for all $p(.) \in C_+(\bar{\Omega})$ and $p^+ < \infty$.*

Proposition 3. *[14] $\forall\ u \in L^{p(.)}(\Omega)$ we have the following relations:*

(i) $\|u\|_{L^{p(.)}(\Omega)} < 1 (= 1; > 1) \Longrightarrow \rho_{p(.)}(u) < 1 (= 1; > 1)$;

(ii) $\|u_n\|_{L^{p(.)}(\Omega)} \longrightarrow 0 (respectively \longrightarrow \infty) \Leftrightarrow \rho_{p(.)}(u_n) \longrightarrow 0 (respectively \longrightarrow \infty)$.

Proposition 4. *[14] $\forall\ u \in W^{1,p(.)}(\Omega)$, we have the following relations:*

(i) $\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|^{p-}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p+}_{1,p(.)}$;

(ii) $\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|^{p+}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p-}_{1,p(.)}$.

Proposition 5. *[6] Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with a smooth boundary and $p \in C_+(\bar{\Omega})$. If $s \in C(\bar{\Omega}, \mathbb{R})$ satisfies the condition $1 \leq s(x) < p^*, \forall x \in \bar{\Omega}$, then the injection $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact, where p^* denotes, as usual, the critical exponent given by*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if\ p(x) < N, \\ \infty & if\ p(x) \geq N. \end{cases}$$

We denote by V the following space:

$$V = \Big\{u \in W^{1,p(\cdot)}(\Omega) : u|_{\partial\Omega} = constant\}$$

and V^* its dual space.

Theorem 2. *[3]*

(i) The space V is a closed subspace of $W^{1,p(.)}(\Omega)$.

(ii) The space V equipped with the norm $\|.\|_{1,p(.)}$ is a reflexive and separable Banach space, $\forall$ $p \in C_+(\bar{\Omega})$ with $p^+ < \infty$.

3 Assumptions on the data

We make the following assumptions on the data:
(H_1) $p : \bar{\Omega} \longrightarrow (1, \infty)$ with $1 < p^- \leq p^+ < \infty$ and p verifies the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad \forall x, y \in \Omega, 0 < |x-y| \leq \frac{1}{2}. \tag{2}$$

For the vector fields $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that
(H_2)

$$A(x, 0) = 0, \tag{3}$$

$\forall$ $x \in \Omega$.
(H_3) There exists a constant C_1 such that

$$|a(x,\xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1}\right), \tag{4}$$

$\forall x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$, with j a positive function in $L^{p'(\cdot)}(\Omega)$.

(H_4) The following monotonicity condition holds:

$$\left[a\left(x,\xi_1\right) - a\left(x,\xi_2\right)\right].\left(\xi_1 - \xi_2\right) > 0, \tag{5}$$

$\forall x \in \Omega$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^N$, with $\xi_1 \neq \xi_2$.
(H_5) The following inequalities holds:

$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi), \tag{6}$$

$\forall x \in \Omega$ and $\forall$ $\xi \in \mathbb{R}^N$.
(H_6) The mapping $a(.,.)$ is odd with respect to its second variable, that is,

$$a(x,-\xi) = -a(x,\xi), \tag{7}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

As examples of models with respect to above assumptions, we can give the following:
(i) If $a(x,\xi) = |\xi|^{p(x)-2}\xi$ then, one has $A(x,\xi) = \dfrac{1}{p(x)}|\xi|^{p(x)}$; which is $p(.)$-Laplacian operator.

(ii) Moreover, if one take $a(x,\xi) = (1+|\xi|^2)^{\frac{(p(x)-2)}{2}}\xi$, one has $A(x,\xi) = \frac{1}{p(x)}\left[(1+|\xi|^2)^{\frac{p(x)}{2}} - 1\right]$; which is the generalized mean curvative operator.

4 WEAK SOLUTION FOR BOUNDED DATA

In this part, we prove the existence and uniqueness of weak solution when $f \in L^{\infty}(\Omega)$.

Definition 1. *A function $u \in V$ is a weak solution to the problem* (1) *if*

$$\int_{\Omega} a(x,\nabla u).\nabla v dx + \int_{\Omega} |u|^{p(x)-2} uv dx - \int_{\Omega} f(x) v dx = 0, \tag{8}$$

$\forall v \in W^{1,p(.)}(\Omega)$.

Definition 2. *The function $I : V \longrightarrow \mathbb{R}$ such that*

$$I(u) = \int_{\Omega} A(x,\nabla u) dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} - \int_{\Omega} f(x) u dx \tag{9}$$

is called the corresponding energy functional of problem (1).

Proposition 6. *The function I is of class $C^1(V,\mathbb{R})$ and its Gâteaux derivative is*

$$\langle I'(u), v\rangle = \int_{\Omega} a(x,\nabla u).\nabla v dx + \int_{\Omega} |u|^{p(x)-2} uv dx - \int_{\Omega} f(x) v dx, \tag{10}$$

$\forall v \in W^{1,p(.)}(\Omega)$.

One therefore sees that, according to above proposition, the critical points of the functional I are the nontrivial weak solutions of problem (1).
The main result of this part is the following:

Theorem 3. *Let $f \in L^{\infty}(\Omega)$. Under the assumptions $(H_1) - (H_6)$, the problem* (1) *has a unique nontrivial weak solution.*

Proof. **A- Proof of the existence of nontrivial weak solutions**
The proof of the existence of weak solutions is deduced from [3], Theorem 1.4.2 .
A- Proof of the uniqueness of a nontrivial weak solution

Let u_1 and u_2 be two weak solutions of problem (1).
$\forall\ v \in W^{1,p(.)}(\Omega)$,

$$\int_{\Omega} a\left(x,\nabla u_1\right).\nabla v dx + \int_{\Omega} |u_1|^{p(x)-2} u_1 v dx = \int_{\Omega} f v dx \tag{11}$$

and

$$\int_{\Omega} a\left(x,\nabla u_2\right).\nabla v dx + \int_{\Omega} |u_2|^{p(x)-2} u_2 v dx = \int_{\Omega} f v dx. \tag{12}$$

Subtracting the two last equalities, one gets

$$\int_{\Omega} a\left(x,\nabla u_1\right).\nabla v dx - \int_{\Omega} a\left(x,\nabla u_2\right).\nabla v dx + \int_{\Omega} |u_1|^{p(x)-2} u_1 v dx - \int_{\Omega} |u_2|^{p(x)-2} u_2 v dx = 0, \tag{13}$$

which is equivalent to say

$$\int_\Omega \left(a\left(x,\nabla u_1\right)-a\left(x,\nabla u_2\right)\right).\nabla v dx+\int_\Omega \left(\left|u_1\right|^{p(x)-2}u_1-\left|u_2\right|^{p(x)-2}u_2\right)v dx=0.$$

By setting $v=u_1-u_2$, one gets

$$\int_\Omega \left(a\left(x,\nabla u_1\right)-a\left(x,\nabla u_2\right)\right).\left(\nabla u_1-\nabla u_2\right)dx+\int_\Omega \left(\left|u_1\right|^{p(x)-2}u_1-\left|u_2\right|^{p(x)-2}u_2\right)\left(u_1-u_2\right)dx=0.$$

Due to the monotonicity of $a(.,.)$ and as $\xi \longmapsto |\xi|^{p(x)-2}\xi$ is monotone, it follows that

$$\int_\Omega \left(a\left(x,\nabla u_1\right)-a\left(x,\nabla u_2\right)\right).\left(\nabla u_1-\nabla u_2\right)dx=0 \tag{14}$$

and

$$\int_\Omega \left(\left|u_1\right|^{p(x)-2}u_1-\left|u_2\right|^{p(x)-2}u_2\right)\left(u_1-u_2\right)dx=0. \tag{15}$$

Since $p^->1$, then, $\forall\ \xi,\ \eta\in\mathbb{R},\ \xi\neq\eta$ (see [10]) one has $\left(|\xi|^{p(x)-2}\xi-|\eta|^{p(x)-2}\eta\right)(\xi-\eta)>0$, $\forall$ $x\in\Omega$. Then, from (15), one deduces that $u_1=u_2$. □

5 ENTROPY SOLUTION FOR INTEGRABLE DATA

In this section, we study problem (1) when the right hand side f belongs in $L^1(\Omega)$. We first recall some auxiliary results. The truncation function of level $t>0$ is defined by

$$T_t(s):=\max\{-t,\min\{t,s\}\}.$$

For any $u\in W^{1,p(.)}(\Omega)$, we denote by $\mathcal{T}^{1,p(.)}(\Omega)$ the set defined by

$$\mathcal{T}^{1,p(.)}(\Omega):=\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }\mid T_t(u)\in W^{1,p(.)}(\Omega),\ t\ >0\}.$$

Proposition 7. *[10] Let $u\in\mathcal{T}^{1,p(.)}(\Omega)$. Then, there exists a unique measurable function $\nu:\Omega\longrightarrow\mathbb{R}^N$ such that $\nabla T_t(u)=\nu\mathcal{X}_{\{|u|\leq t\}}$, $\forall\ t>0$. The function ν is denoted by ∇u. In addition, if $u\in W^{1,p(.)}(\Omega)$, then $\nu\in(L^{p(.)}(\Omega))^N$ and $\nu=\nabla u$ in the usual sense.*

We define $\mathcal{T}^{1,p(.)}_{tr}(\Omega)$ as the set of function $u\in\mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $u_n\subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:
(C_1) : $u_n\longrightarrow u$ a.e. in Ω.
(C_2) : $\nabla T_t(u_n)\longrightarrow\nabla T_t(u)$ in $L^1(\Omega)$, $\forall\ t>0$.
(C_3): $u_n\longrightarrow$ *constant* a.e. on $\partial\Omega$.

Definition 3. *A measurable function u is an entropy solution of problem* (1) *if $u\in\mathcal{T}^{1,p(.)}_{tr}(\Omega)$ and* $\forall\ t>0$,

$$\int_\Omega a(x,\nabla u).\nabla T_t(u-v)dx+\int_\Omega |u|^{p(x)-2}uT_t(u-v)dx\leq\int_\Omega f(x)T_t(u-v)dx, \tag{16}$$

$\forall v\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

Remark 1. *If (H_5) is satisfied then,*

$$\begin{aligned}\int_\Omega a(x,\nabla u).\nabla T_t(u)dx &= \int_{\{|u|\leq t\}} a(x,\nabla u).\nabla T_t(u)dx\\ &= \int_{\{|u|\leq t\}} a(x,\nabla T_t(u)).\nabla T_t(u)dx\\ &\geq \int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}dx \ \geq 0.\end{aligned}$$

Note also that

$$\begin{aligned}\int_\Omega |u|^{p(x)-2}uT_t(u)dx &= \int_{\{|u|\leq t\}} |u|^{p(x)-2}uT_t(u)dx + \int_{\{|u|>t\}} |u|^{p(x)-2}uT_t(u)dx\\ &= \int_{\{|u|\leq t\}} |u|^{p(x)-2}u^2dx + \int_{\{|u|>t\}} |u|^{p(x)-2}ut.\frac{u}{|u|}dx\\ &= \int_{\{|u|\leq t\}} |u|^{p(x)}dx + t\int_{\{|u|>t\}} |u|^{p(x)-1}dx \geq 0, \ \forall\, t>0.\end{aligned}$$

Which means that

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx \geq 0 \text{ and } \int_\Omega |u|^{p(x)-2}uT_t(u)dx \geq 0.$$

We deduce from (16) *by taking $v=0$ that*

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx \leq \int_\Omega f(x)T_t(u)dx \tag{17}$$

and

$$\int_\Omega |u|^{p(x)-2}uT_t(u)dx \leq \int_\Omega f(x)T_t(u)dx. \tag{18}$$

Theorem 4. *Let $f\in L^1(\Omega)$. Suppose that hypotheses $(H_1)-(H_6)$ holds. Then, problem* (1) *admits a unique entropy solution.*

For the proof of the Theorem 4, we need the following results.

Proposition 8. *Let $f\in L^1(\Omega)$. Suppose that hypotheses $(H_1)-(H_6)$ holds. If u is an entropy solution of* (1), *then*

$$\int_\Omega |\nabla T_t(u)|^{p(x)}\,dx \leq t\|f\|_{L^1(\Omega)},\ \forall\, t>0 \tag{19}$$

and

$$\||u|^{p(x)-2}u\|_{L^1(\Omega)} = \||u|^{p(x)-1}\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{20}$$

Proof. Using (17) we have

$$\begin{aligned}\int_\Omega a(x,\nabla u).\nabla T_t(u)dx &\leq \int_\Omega f(x)T_t(u)dx\\ &\leq t\|f\|_{L^1(\Omega)}.\end{aligned} \tag{21}$$

Moreover, according to (H_5),

$$\int_\Omega a(x,\nabla u).\nabla T_t(u)dx = \int_{\{|u|\leq t\}} a(x,\nabla T_t(u)).\nabla T_t(u)dx \geq \int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}dx. \tag{22}$$

From (21) and (22) we deduce that

$$\int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}dx \leq t\|f\|_{L^1(\Omega)},\ \forall\, t\ >\ 0;$$

which is equivalent to say

$$\int_{\Omega} |\nabla T_t(u)|^{p(x)}\, dx \leq t\|f\|_{L^1(\Omega)},\ \forall\, t > 0.$$

One has according to (18)

$$\int_{\Omega} |u|^{p(x)-2}uT_t(u)dx \leq \int_{\Omega} f(x)T_t(u)dx \leq t\|f\|_{L^1(\Omega)}. \tag{23}$$

From (23), one obtains

$$\int_{\Omega\cap\{|u|\geq t\}} |u|^{p(x)-2}uT_t(u)dx \leq t\|f\|_{L^1(\Omega)}.$$

Which is equivalent to say

$$\int_{\Omega\cap\{u\geq t\}} t|u|^{p(x)-2}udx - \int_{\Omega\cap\{u\leq -t\}} t|u|^{p(x)-2}udx \leq t\|f\|_{L^1(\Omega)}.$$

Therefore,

$$\int_{\Omega\cap\{|u|\geq t\}} |u|^{p(x)-1}dx \leq \|f\|_{L^1(\Omega)}. \tag{24}$$

Finally, as $t \longrightarrow 0$ in (24), it follows by using Fatou's Lemma that

$$\||u|^{p(x)-2}u\|_{L^1(\Omega)} = \||u|^{p(x)-1}\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}.$$

□

Proposition 9. *Let $f \in L^1(\Omega)$. Suppose that hypotheses $(H_1)-(H_6)$ holds and let u be an entropy solution of* (1). *If there exists a positive constant M such that*

$$\int_{\{|u|>t\}} t^{q(x)}dx \leq M,$$

$\forall t > 0$, then,

$$\int_{\{|\nabla u^{\alpha(\cdot)}|>t\}} t^{q(x)}dx \leq M + \|f\|_{L^1(\Omega)},$$

$\forall t > 0$, where $\alpha(.) = \frac{p(.)}{(q(.)+1)}$ and $q(.) : \bar{\Omega} \longrightarrow (0,\infty)$ is measurable such that $q^- > 0$.

Proof.

$$\begin{aligned}
\int_{\{|\nabla u|^{\alpha(.)}>t\}} t^{q(x)}dx &= \int_{\{|\nabla u|^{\alpha(.)}>t,|u|\leq t\}} t^{q(x)}dx + \int_{\{|\nabla u|^{\alpha(.)}>t,|u|>t\}} t^{q(x)}dx \\
&\leq \int_{\{|u|\leq t\}} t^{q(x)} \left(\frac{|\nabla u|^{\alpha(.)}}{t}\right)^{\frac{p(x)}{\alpha(x)}} dx + \int_{\{|u|>t\}} t^{q(x)}dx \\
&\leq \frac{1}{t}\int_{\{|u|\leq t\}} |\nabla u|^{p(x)}dx + M \\
&\leq \frac{1}{t}(t\|f\|_{L^1(\Omega)}) + M \\
&\leq \|f\|_{L^1(\Omega)} + M.
\end{aligned}$$

□

Proposition 10. *Let u be an entropy solution to problem* (1)*. Then, $meas\{|u| > t\}$ and $meas\{|\nabla u| > t\}$ tend toward 0 as $t \to \infty$.*

Proof. The proof is done in a similar way as in the proof of Proposition 4.7 in [10]. Indeed, as $u = constant$ on $\partial\Omega$, then one can uses $T_t(constant)$ instead of $T_t(u)$ on $\partial\Omega$ which is a constant in the proof. □

Proof of Theorem 4.

A- Proof of the uniqueness of the entropy solution

Let $k > 0$, u_1 and u_2 two entropy solutions of problem (1).

One has

$$\begin{cases} \displaystyle\int_{\{|u_1 - T_k(u_2)| \le t\}} a(x, \nabla u_1).\nabla T_t(u_1 - T_k(u_2))\,dx + \int_\Omega |u_1|^{p(x)-2} u_1 T_t(u_1 - T_k(u_2))\,dx \le \\ \displaystyle\int_\Omega f(x) T_t(u_1 - T_k(u_2))\,dx. \end{cases} \tag{25}$$

and

$$\begin{cases} \displaystyle\int_{\{|u_2 - T_k(u_1)| \le t\}} a(x, \nabla u_2).\nabla T_t(u_2 - T_k(u_1))\,dx + \int_\Omega |u_2|^{p(x)-2} u_2 T_t(u_2 - T_k u_1)\,dx \le \\ \displaystyle\int_\Omega f(x) T_t(u_2 - T_k(u_1))\,dx. \end{cases} \tag{26}$$

By adding (25) and (26), one obtains

$$\begin{cases} \displaystyle\int_{\{|u_1 - T_k(u_2)| \le t\}} a(x, \nabla u_1).\nabla T_t(u_1 - T_k(u_2))\,dx + \\ \displaystyle\int_{\{|u_2 - T_k(u_1)| \le t\}} a(x, \nabla u_2).\nabla T_t(u_2 - T_k(u_1))\,dx + \int_\Omega |u_1|^{p(x)-2} u_1 T_t(u_1 - T_k(u_2))\,dx + \\ \displaystyle\int_\Omega |u_2|^{p(x)-2} u_2 T_t(u_2 - T_k(u_1))\,dx \le \int_\Omega f(x)\left(T_t(u_1 - T_k(u_2)) + T_t(u_2 - T_k(u_1))\right)dx. \end{cases} \tag{27}$$

Let us define the following sets:

$A_1 := \{|u_1 - u_2| \le t\}$, $A_2 := \{|u_1| \le k\}$, $A_3 := \{|u_2| \le k\}$, $A_4 := \{|u_1| > k\}$, $A_5 := \{|u_2| > k\}$.

One has

$$\begin{aligned} \int_{\{|u_1 - T_k(u_2)| \le t\}} a(x, \nabla u_1).\nabla T_t(u_1 - T_k(u_2))\,dx \ge & \int_{A_1 \cap A_3 \cap A_2} a(x, \nabla u_1).\nabla(u_1 - u_2)\,dx \\ & - \int_{A_1 \cap A_3 \cap A_4} a(x, \nabla u_1).\nabla u_2 dx \end{aligned} \tag{28}$$

Using (H_3) and Hölder type inequality, it follows that

$$\begin{cases} \displaystyle\left|\int_{A_1 \cap A_3 \cap A_4} a(x, \nabla u_1).\nabla u_2 dx\right| \le C_1 \int_{A_1 \cap A_3 \cap A_4} \left(j(x) + |\nabla u_1|^{p(x)-1}\right) |\nabla u_2|\,dx \\ \displaystyle\le C_1 \left(\|j\|_{L^{p'(\cdot)}(A_1 \cap A_3 \cap A_4)} + \left\||\nabla u_1|^{p(x)-1}\right\|_{L^{p'(\cdot)}(A_1 \cap A_3 \cap A_4)}\right) \|\nabla u_2\|_{L^{p(\cdot)}(A_1 \cap A_3 \cap A_4)}. \end{cases} \tag{29}$$

According to Proposition 10 (see[10]),

$$C_1 \left(\|j\|_{L^{p'(\cdot)}(A_1 \cap A_3 \cap A_4)} + \left\||\nabla u_1|^{p(x)-1}\right\|_{L^{p'(\cdot)}(A_1 \cap A_3 \cap A_4)}\right) \|\nabla u_2\|_{L^{p(\cdot)}(A_1 \cap A_3 \cap A_4)}$$

converges to 0 as $k \longrightarrow \infty$. Therefore,

$$\int_{\{|u_1-T_k(u_2)|\le t\}} a\left(x,\nabla u_1\right).\nabla T_t\left(u_1-T_k(u_2)\right)dx \ge \varepsilon_1(k)+\int_{A_1\cap A_3\cap A_2} a\left(x,\nabla u_1\right).\nabla\left(u_1-u_2\right)dx, \tag{30}$$

with $\lim_{k\longrightarrow\infty}\varepsilon_1(k)=0$.
By an analogous reasoning, one also deduces that

$$\int_{\{|u_2-T_k(u_1)|\le t\}} a\left(x,\nabla u_2\right).\nabla T_t\left(u_2-T_k(u_1)\right)dx \ge \varepsilon_2(k)-\int_{A_1\cap A_3\cap A_2} a\left(x,\nabla u_2\right).\nabla\left(u_1-u_2\right)dx, \tag{31}$$

with $\lim_{k\longrightarrow\infty}\varepsilon_2(k)=0$.
One also has $|u_1|^{p(x)-2}u_1(T_t(u_1-T_k(u_2))\longrightarrow|u_1|^{p(x)-2}u_1T_t(u_1-u_2)$ a.e. in Ω as $k\longrightarrow\infty$.
Moreover, $\left||u_1|^{p(x)-2}u_1(T_t(u_1-T_k(u_2))\right|\le t|u_1|^{p(x)-1}\in L^1(\Omega)$.
Then, by the Lebesgue dominated convergence Theorem one deduces that

$$\lim_{k\longrightarrow\infty}\int_\Omega|u_1|^{p(x)-2}u_1(T_t(u_1-T_k(u_2))dx=\int_\Omega|u_1|^{p(x)-2}u_1T_t(u_1-u_2)dx. \tag{32}$$

By a similar reasoning, one deduces that

$$\lim_{k\longrightarrow\infty}\int_\Omega|u_2|^{p(x)-2}u_2(T_t(u_2-T_k(u_1))dx=\int_\Omega|u_2|^{p(x)-2}u_2T_t(u_2-u_1)dx. \tag{33}$$

Thus by using (32) and (33), one obtains

$$\begin{cases}\lim\limits_{k\longrightarrow+\infty}\left(\int_\Omega|u_1|^{p(x)-2}u_1T_t(u_1-T_k(u_2)dx+\int_\Omega|u_2|^{p(x)-2}u_2T_t(u_2-T_k(u_1)dx\right)\\ =\int_\Omega|u_1|^{p(x)-2}u_1T_t(u_1-u_2)dx+\int_\Omega|u_2|^{p(x)-2}u_2T_t(u_2-u_1)dx.\end{cases} \tag{34}$$

$\forall\ t>0$, one has

$$f(x)\Big(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\Big)\longrightarrow f(x)\Big(T_t(u_1-u_2)+T_t(u_2-u_1)\Big)=0$$

a.e. in Ω as $k\longrightarrow\infty$. In addition,

$$\left|f(x)\Big(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))\Big)\right|\le 2t|f|\in L^1(\Omega).$$

Thus, by the Lebesgue dominated convergence Theorem, one has

$$\lim_{k\longrightarrow\infty}\int_\Omega f(x)\Big(T_t(u_1-T_k(u_2))+T_t(u_2-T_k(u_1))dx\Big)=0. \tag{35}$$

Therefore, by using (27), (30), (31) and (34), it follows that

$$\left\{\begin{array}{l}\displaystyle\int_{\{|u_1-T_k(u_2)|\leq t\}} a\left(x,\nabla u_1\right).\nabla T_t\left(u_1-T_k(u_2)\right)dx \\ \displaystyle\qquad + \int_{\{|u_2-T_k(u_1)|\leq t\}} a\left(x,\nabla u_2\right).\nabla T_t\left(u_2-T_k(u_1)\right)dx \\ \displaystyle\qquad + \int_\Omega |u_1|^{p(x)-2}u_1 T_t\left(u_1-T_k(u_2)\right)dx \\ \displaystyle\qquad + \int_\Omega |u_2|^{p(x)-2}u_2 T_t\left(u_2-T_k(u_1)\right)dx \\ \displaystyle\qquad \geq \int_{A_1\cap A_3\cap A_2} a\left(x,\nabla u_1\right).\nabla\left(u_1-u_2\right)dx - \int_{A_1\cap A_3\cap A_2} a\left(x,\nabla u_2\right).\nabla\left(u_1-u_2\right)dx \\ \displaystyle\qquad + \int_\Omega\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)dx+\varepsilon_1(k)+\varepsilon_2(k).\end{array}\right. \tag{36}$$

From (27), (35) and (36), one deduces, as $k\longrightarrow\infty$, that

$$\int_{A_1\cap A_2\cap A_3}\left[a(x,\nabla u_1)-a(x,\nabla u_2)\right].\nabla(u_1-u_2)dx+\int_\Omega\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)dx\leq 0. \tag{37}$$

Therefore,

$$\int_{A_1\cap A_2\cap A_3}\left[a(x,\nabla u_1)-a(x,\nabla u_2)\right].\nabla(u_1-u_2)dx=0 \tag{38}$$

and

$$\int_\Omega\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)dx=0, \tag{39}$$

since the two terms in (37) are positive. From (38) and (H_4), it follows that

$$u_1-u_2=c\ a.e\ in\ \Omega\ , \tag{40}$$

where c is real constant. By (39), we deduce that for all $k\geq 1$ there exists $C_k\subset\Omega$, $meas(C_k)=0$ such that for all $x\in\Omega\setminus C_k$,

$$\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)T_t\left(u_1-u_2\right)=0.$$

Therefore,

$$\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)\left(u_1-u_2\right)=0, \tag{41}$$

for all $x\in\Omega\setminus\bigcup_{k\geq 1}C_k$.
Since $p^->1$, then

$$\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)\left(u_1-u_2\right)>0,$$

one has, by using (41),

$$u_1-u_2=0\ a.e\ in\ \Omega. \tag{42}$$

Finally, (40) and (42) give

$$u_1=u_2\ \ a.e\ in\ \Omega.$$

B-Proof of the existence of entropy solution
Let $(f_n)_{n\in\mathbb{N}}\subset L^\infty(\Omega)$ defined by $f_n=T_n(f)$. One has

$$\|f_n\|_1\leq\|f\|_1,\quad\forall\quad n\in\mathbb{N}. \tag{43}$$

We consider the following approximate problem:

$$\begin{cases} -\operatorname{div}\left(a\left(x,\nabla u_n\right)\right)+\left|u_n\right|^{p(x)-2}u_n=f_n \text{ in } \Omega, \\ u_n=constant \qquad\qquad\qquad\qquad on\ \partial\Omega. \end{cases} \tag{44}$$

According to Theorem 3, for any $n\in\mathbb{N}$, problem (44) has a unique weak solution $u_n\in V$.
Our objective is to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ of weak solutions of problem (44) converges to a measurable function u which is the entropy solution of problem (1).
We begin by proving that the sequence of weak solutions of problem (44) converges in measure to a measurable function u.

Lemma 1. $\forall\ t>0$, $\|T_t(u_n)\|_{1,p(.)}(\Omega)\le 1+C$ *where C is a positive constant.*

Proof. One has $\forall\ n\in\mathbb{N}$,

$$\int_\Omega a(x,\nabla u_n).\nabla v dx+\int_\Omega |u_n|^{p(x)-2}u_n v dx=\int_\Omega f_n(x)v dx, \tag{45}$$

$\forall\ v\in W^{1,p(.)}(\Omega)$.
By choosing $v=T_t(u_n)$ in the equality above, one obtains

$$\int_\Omega a(x,\nabla u_n).\nabla T_t(u_n)dx+\int_\Omega |u_n|^{p(x)-2}u_n T_t(u_n)dx=\int_\Omega f_n(x)T_t(u_n)dx.$$

Then,

$$\int_\Omega a(x,\nabla u_n).\nabla T_t(u_n)dx\le\int_\Omega f_n(x)T_t(u_n)dx\le t\|f_n\|_{L^1(\Omega)}\le t\|f\|_{L^1(\Omega)}.$$

Therefore,

$$\int_\Omega a(x,\nabla T_t(u_n)).\nabla T_t(u_n)dx\le t\|f\|_{L^1(\Omega)}.$$

Thus,

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}dx\le t\|f\|_{L^1(\Omega)}. \tag{46}$$

$$\int_\Omega |T_t(u_n)|^{p(x)}dx=\int_{\{|u_n|\le t\}}|T_t(u_n)|^{p(x)}dx+\int_{\{|u_n|>t\}}|T_t(u_n)|^{p(x)}dx. \tag{47}$$

Moreover,

$$\begin{aligned}\int_{\{|u_n|>t\}}|T_t(u_n)|^{p(x)}dx &\le \int_{\{|u_n|>t,t>1\}}t^{p(x)}dx+\int_{\{|u_n|>t,t\le 1\}}t^{p(x)}dx\\ &\le \int_{\{|u_n|>t,t>1\}}t^{p^+}dx+meas(\Omega)\\ &\le meas(\Omega)t^{p^+}+meas(\Omega)\\ &\le meas(\Omega)(t^{p^+}+1).\end{aligned}$$

One deduces that

$$\int_{\{|u_n|>t\}}|T_t(u_n)|^{p(x)}dx\le meas(\Omega)(t^{p^+}+1). \tag{48}$$

Therefore

$$\begin{aligned}\int_{\{|u_n|\leq t\}}|T_t(u_n)|^{p(x)}dx &\leq \int_{\{|u_n|\leq t\}}t^{p(x)}dx\\ &\leq \int_{\{|u_n|\leq t, t>1\}}t^{p(x)}dx+\int_{\{|u_n|\leq t, t\leq 1\}}t^{p(x)}dx\\ &\leq \int_{\{|u_n|\leq t, t>1\}}t^{p^+}dx+meas(\Omega)\\ &\leq meas(\Omega)t^{p^+}+meas(\Omega)\\ &\leq meas(\Omega)(t^{p^+}+1).\end{aligned}$$

One deduces that

$$\int_{\{|u_n|\leq t\}}|T_t(u_n)|^{p(x)}dx\leq meas(\Omega)(t^{p^+}+1). \tag{49}$$

From (48) and (49), one obtains

$$\int_{\Omega}|T_t(u_n)|^{p(x)}dx\leq 2meas(\Omega)(t^{p^+}+1). \tag{50}$$

From (46) and (50), one deduces that

$$\rho_{1,p(.)(\Omega)}(T_t(u_n))\leq t\|f\|_{L^1(\Omega)}+2(1+t^{p^+})meas(\Omega)=const(t,f,p^+,meas(\Omega)). \tag{51}$$

For $\|T_t(u_n)\|^{p^-}_{1,p(.)(\Omega)}\geq 1$, one has

$$\|T_t(u_n)\|^{p^-}_{1,p(.)(\Omega)}\leq \rho_{W^{1,p(.)}(\Omega)}(T_t(u_n))\leq const(t,f,p^+,meas(\Omega)). \tag{52}$$

Using (52), one obtains

$$\|T_t(u_n)\|_{1,p(.)(\Omega)}\leq \left[const(t,f,p^+,meas(\Omega))\right]^{\frac{1}{p^-}}.$$

Therefore,

$$\|T_t(u_n)\|_{1,p(.)(\Omega)}\leq 1+\left[const(t,f,p^+,meas(\Omega))\right]^{\frac{1}{p^-}}.$$

Hence,

$$\|T_t(u_n)\|_{1,p(.)(\Omega)}\leq 1+C.$$

□

Remark 2. *One has $\forall\, t>0$, $\|T_t(u_n)\|_{1,p(.)(\Omega)}\leq 1+C$. So, the sequence $(T_t(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p(.)}(\Omega)$ and therefore in $L^{p^-}(\Omega)$. Consequently, there exists a subsequence still denoted by $(T_t(u_n))_{n\in\mathbb{N}}$ converging strongly to σ_t in $L^{p^-}(\Omega)$.*

Proposition 11. *Let u_n be a weak solution to problem (44). Then, $meas\{|u_n|>t\}$ and $meas\{|\nabla u_n|>t\}$ tend toward 0 as $t\to\infty$.*

Proof. From (46), one has

$$\int_{\Omega}|\nabla T_t(u_n)|^{p(x)}dx\leq t\|f\|_{L^1(\Omega)}.$$

By analogous raisonning as in the proof of the Proposition 10, one gets $meas\{|u_n|>t\}$ and $meas\{|\nabla u_n|>t\}$ tend toward 0 as $t\to\infty$. □

Proposition 12. *Let $f \in L^1(\Omega)$. Suppose that hypotheses $(H_1) - (H_6)$ holds. Let $(u_n)_{n\in\mathbb{N}} \subset V$ be the solutions sequence corresponding to $(f_n)_{n\in\mathbb{N}}$ then, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Therefore, there exists a measurable function u and a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that $u_n \longrightarrow u$ in measure.*

Proof. The proof consists to show that

$$\forall\delta > 0, \forall\varepsilon > 0, \exists n_0 \in \mathbb{N}/\forall n \geq n_0, \forall m \geq n_0, \text{meas}\{|u_n - u_m| \geq \delta\} \leq \varepsilon.$$

Let's put $A_1 = \{|u_n| > t\}, A_2 = \{|u_m| > t\}, A_3 = \{|T_t(u_n) - T_t(u_m)| > \delta\}, A_4 = \{|u_n - u_m| > \delta\}.$

We see that $A_4 \subset A_1 \cup A_2 \cup A_3$, therefore, one obtains

$$\text{meas}(A_4) \leq \text{meas}(A_1) + \text{meas}(A_2) + \text{meas}(A_3). \tag{53}$$

By Proposition 11, one deduces that

$$meas(A_1) \leq \frac{\varepsilon}{3} \tag{54}$$

and

$$meas(A_2) \leq \frac{\varepsilon}{3}. \tag{55}$$

Acording to Remark 2, $(T_t(u_n))_{n\in\mathbb{N}}$ converges strongly in $L^{p^-}(\Omega)$, so it is a Cauchy sequence in measure in $L^{p^-}(\Omega)$. Therefore,

$$meas(A_3) \leq \int_\Omega \frac{|T_t(u_n) - T_t(u_m)|^{p^-}}{\delta^{p^-}} dx \leq \frac{\varepsilon}{3}, \tag{56}$$

$\forall\ n,\ m \geq n_0$.

From (53), (54), (55) and (56), one deduces that

$$meas(A_4) \leq \varepsilon.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is therefore a Cauchy sequence in measure. So there exists a measurable function u such that $u_n \longrightarrow u$ in measure, up to a subsequence, one can assume that $u_n \longrightarrow u$ a.e. in Ω. □

Lemma 2. *[14] Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. If $(u_n)_{n\in\mathbb{N}}$ converges in measure to u and is uniformly bounded in $L^{p(\cdot)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then $u_n \longrightarrow u$ strongly in $L^1(\Omega)$.*

Lemma 3. *[7] Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Consider a measurable function $\gamma : X \longrightarrow [0, \infty)$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \varepsilon, \qquad for\ all\ A \in \mathcal{M}\ with\ \int_A \gamma d\mu < \delta.$$

Lemma 4. *[10]*

(a) For a.e. $t \in \mathbb{R}$, $\nabla T_t(u_n)$ converges in measure to $\nu\mathcal{X}_{\{|u|<t\}}$.

(b) For a.e. $t \in \mathbb{R}$, $\nabla T_t(u) = \nu\mathcal{X}_{\{|u|<t\}}$.

(c) $\nabla T_t(u) = \nu \mathcal{X}_{\{|u|<t\}}, \forall\, t \in \mathbb{R}$.

Let $u_n \in V$ be a weak solution of problem (45). Then, taking $T_t\,(u_n - v)$ as test function in (8), one obtains

$$\int_\Omega a(x,\nabla u).\nabla\left(T_t\,(u_n - v)\right)dx + \int_\Omega |u_n|^{p(x)-2}\,u_n\left(T_t\,(u_n - v)\right)dx - \int_\Omega f_n(x)\left(T_t\,(u_n - v)\right)dx = 0, \tag{57}$$

for all $v \in W^{1,p(.)}(\Omega)$.
one has the following results.

Proposition 13. *Let $f \in L^1(\Omega)$. Suppose that hypotheses $(H_1) - (H_6)$ holds. Let u_n be a weak solution of problem (44), then:*
(i) $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to the weak gradient of u.
(ii) For all $t > 0$, $\nabla T_t(u_n)$ converges to $\nabla T_t(u)$ in $(L^1(\Omega))^N$.
(iii) $a(x, \nabla T_t(u_n))$ converges to $a(x, \nabla T_t(u))$ strongly in $\left(L^1(\Omega)\right)^N$ and weakly in $\left(L^{p'(.)}(\Omega)\right)^N$.
(iv) $u_n \longrightarrow$ constant a.e on $\partial\Omega$.

Proof.
• Proof of (i)
The proof consists to show that $\forall\, \delta > 0, \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}\ /\ \forall n \geq n_0$,
$\forall m \geq n_0$ meas $\{|\nabla u_n - \nabla u_m| \geq \delta\} \leq \varepsilon$.
Let's put $A_1 = \{|\nabla u_n| > k\}, A_2 = \{|\nabla u_m| > k\}, A_3 = \{|u_n - u_m| > t\}$,
$A_4 = \{|\nabla u_n - \nabla u_m| \geq \delta, |\nabla u_n| \leq k, |\nabla u_m| \leq k, |u_n - u_m| \leq t\}$.
Then

$$\{|\nabla u_n - \nabla u_m| \geq \delta\} \subset A_1 \cup A_2 \cup A_3 \cup A_4.$$

Thus,

$$\text{meas}\{|\nabla u_n - \nabla u_m| \geq \delta\} \leq \text{meas}(A_1) + \text{meas}(A_2) + \text{meas}(A_3) + \text{meas}(A_4). \tag{58}$$

By Proposition 11, one has

$$meas(A_1) \leq \frac{\varepsilon}{4} \tag{59}$$

and

$$meas(A_2) \leq \frac{\varepsilon}{4}. \tag{60}$$

According to Proposition 12, one has

$$meas(A_3) \leq \frac{\varepsilon}{4}. \tag{61}$$

Let's now estimate meas (A_4).
One has from (H_4),

$$\left[a\,(x, \xi_1) - a\,(x, \xi_2)\right].(\xi_1 - \xi_2) > 0, \text{ if } \xi_1 - \xi_2 \neq 0.$$

Moreover, since $a(x, \xi)$ is continuous with respect to ξ for a.e $x \in \Omega$, then there exists a real valued function $\gamma : \Omega \longrightarrow [0, \infty)$ such that $meas(\{x \in \Omega : \gamma(x) = 0\}) = 0$ and

$$\left[a\,(x, \xi_1) - a\,(x, \xi_2)\right].(\xi_1 - \xi_2) > \gamma(x), \tag{62}$$

for all ξ_1, $\xi_2 \in \mathbb{R}^N$ such that $|\xi_1| \leq k, |\xi_2| \leq k, |\xi_1 - \xi_2| \geq \delta$ for a.e $x \in \Omega$.
Let $\delta = \delta(\varepsilon)$ be given by Lemma 3, replacing ε and A by $\frac{\varepsilon}{4}$ and A_4 respectively, one has, by taking $T_t(u_n - u_m)$ as test function in (45), that

$$\begin{aligned}\int_\Omega a(x, \nabla u_n).\nabla T_t(u_n - u_m)dx + \int_\Omega |u_n|^{p(x)-2} u_n T_t(u_n - u_m)dx &= \int_\Omega f_n T_t(u_n - u_m)dx \\ &\leq t\|f\|_{L^1(\Omega)}.\end{aligned} \tag{63}$$

In an analogous way, taking $T_t(u_m - u_n)$ as test function in (45), one has

$$\begin{aligned}\int_\Omega a(x, \nabla u_m).\nabla T_t(u_m - u_n)dx + \int_\Omega |u_m|^{p(x)-2} u_m T_t(u_m - u_n)dx &= \int_\Omega f_n T_t(u_m - u_n)dx \\ &\leq t\|f\|_{L^1(\Omega)}.\end{aligned} \tag{64}$$

By adding (63) and (64), it follows

$$\begin{aligned}\int_{\{|u_n - u_m|\}\leq t} [a(x, \nabla u_n) - a(x, \nabla u_m)] .(\nabla u_n - \nabla u_m)dx \quad &+ \quad \int_\Omega (|u_n|^{p(x)-2} u_n - |u_m|^{p(x)-2} u_m) T_t(u_n - u_m)dx \\ &\leq \quad 2t\|f\|_{L^1(\Omega)}.\end{aligned}$$

Then,

$$\begin{aligned}\int_{A_4} \gamma dx \quad &\leq \quad \int_{A_4} [a(x, \nabla u_n) - a(x, \nabla u_m)].(\nabla u_n - \nabla u_m)dx \\ &\leq \quad 2t\|f\|_{L^1(\Omega)} \leq \delta,\end{aligned}$$

where $t = \dfrac{\delta}{2\|f\|_{L^1(\Omega)}}$.
One obtains by using Lemma 3,

$$meas(A_4) \leq \frac{\varepsilon}{4}. \tag{65}$$

By combining (59), (60), (61) and (65), one deduces from (58) that meas $\{|\nabla u_n - \nabla u_m| \geq \delta\} \leq \varepsilon$. Therefore, the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure.
Thus, the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to a measurable function ν. From Lemma 4, one can concludes that $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to the weak gradient of u. Hence (i) is proved.
• Proof of (ii)
One first shows that $\nabla T_t(u_n)$ is a Cauchy sequence in measure. The proof consists to show that $\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}/\forall n \geq n_0, \forall m \geq n_0$ maes $\{|\nabla T_t(u_n) - \nabla T_t(u_m)| \geq \delta\} \leq \varepsilon$. Let's put $A_1 = \{|\nabla u_n - \nabla u_m| > \delta, |u_n| \leq t, |u_m| \leq t\}$; $A_2 = \{|\nabla u_n| > \delta, |u_n| > t, |u_m| \leq t\}$;
$A_3 = \{|\nabla u_n| > \delta, |u_m| > t|u_n| \leq t\}$; $A_4 = \{|u_n| > t, |u_m| > t\}$.
Note that

$$\{|\nabla T_t(u_n) - \nabla T_t(u_m)| \geq \delta\} \subset A_1 \cup A_2 \cup A_3 \cup A_4. \tag{66}$$

By Proposition 11, one obtains

$$meas(A_2) \leq \frac{\varepsilon}{4}, \qquad meas(A_3) \leq \frac{\varepsilon}{4}, \qquad and \qquad meas(A_4) \leq \frac{\varepsilon}{4}. \tag{67}$$

Using Proposition 13-(i), one has

$$meas(A_1) \leq \frac{\varepsilon}{4}. \tag{68}$$

Therefore, using (66), (67) and (68), one gets

$$\text{meas}\,\{|\nabla T_t(u_n) - \nabla T_t(u_m)| \geq \delta\} \leq \varepsilon. \tag{69}$$

Thus, $\nabla T_t(u_n)$ is a Cauchy sequence in measure. Consequently, $\nabla T_t(u_n)$ converges in measure to $\nabla T_t(u)$. Then, using lemmas 1 and 2, (ii) follows.

• Proof of (iii)

According to lemmas 2 and 4, we know that $\forall\, t > 0$, $a(x, \nabla T_t(u_n))$ converges strongly to $a(x, \nabla T_t(u))$ in $\left(L^1(\Omega)\right)^N$ and converges weakly to $\mathcal{X}_t \in \left(L^{p'(\cdot)}(\Omega)\right)^N$, which means that the sequence $a(x, \nabla T_t(u_n))_{n\in\mathbb{N}}$ converges weakly in $\left(L^1(\Omega)\right)^N$ and therefore in $\mathcal{X}_t = a(x, \nabla T_t(u))$. Thus, $a(x, \nabla T_t(u)) \in \left(L^{p'(\cdot)}(\Omega)\right)^N$.

• Proof of (iv).

One has

$$\begin{aligned}\int_\Omega |T_t(u_n)|^{p^-}dx &= \int_{\{|T_t(u_n)|\le 1\}} |T_t(u_n)|^{p^-}dx + \int_{\{|T_t(u_n)|>1\}} |T_t(u_n)|^{p^-}dx\\ &\le meas(\Omega) + \int_{\{|T_t(u_n)|>1\}} |T_t(u_n)|^{p(x)}dx\\ &\le meas(\Omega) + \int_\Omega |T_t(u_n)|^{p(x)}dx.\end{aligned}$$

By using (50), one obtains

$$\int_\Omega |T_t(u_n)|^{p^-}dx \le meas(\Omega)(3 + 2t^{p^+}) \tag{70}$$

and

$$\begin{aligned}\int_\Omega |\nabla T_t(u_n)|^{p^-}dx &= \int_{\{|\nabla T_t(u_n)|\le 1\}} |\nabla T_t(u_n)|^{p^-}dx + \int_{\{|\nabla T_t(u_n)|>1\}} |\nabla T_t(u_n)|^{p^-}dx\\ &\le meas(\Omega) + \int_{\{|\nabla T_t(u_n)|>1\}} |\nabla T_t(u_n)|^{p(x)}dx\\ &\le meas(\Omega) + \int_\Omega |\nabla T_t(u_n)|^{p(x)}dx.\end{aligned}$$

One obtains by using (47),

$$\int_\Omega |\nabla T_t(u_n)|^{p^-}dx \le meas(\Omega) + t\|f\|_{L^1(\Omega)}. \tag{71}$$

For all $1 \le p^- \le \infty$,

$$\tau : W^{1,p^-}(\Omega) \longrightarrow L^{p^-}(\partial\Omega); u \longmapsto \tau(u) = u|_{\partial\Omega}$$

is compact.

As $T_t(u_n) \rightharpoonup T_t(u)$ in $W^{1,p^-}(\Omega)$, one deduces that $T_t(u_n)$ converges strongly to $T_t(u)$ in $L^{p^-}(\partial\Omega)$. Therefore, there exists a subsequence still denoted $(T_t(u_n))$ which converges to $T_t(u)$ a.e. on $\partial\Omega$. In other words, there exists $C \subset \partial\Omega$ such that $T_t(u_n)$ converges to $T_t(u)$ on $\partial\Omega\backslash C$ with $\sigma(C) = 0$ where σ refers to the surface measurement on $\partial\Omega$.

By using (70), (71) and Hölder's inequality, one obtains

$$\int_\Omega |T_t(u_n)|dx \le (meas(\Omega))^{\frac{1}{(p^-)^*}}\left(meas(\Omega)(3 + 2t^{p^+})\right)^{\frac{1}{(p^-)^*}} \tag{72}$$

and

$$\int_\Omega |\nabla T_t(u_n)|dx \le (meas(\Omega))^{\frac{1}{(p^-)^*}}\left(meas(\Omega) + t\|f\|_{L^1(\Omega)}\right)^{\frac{1}{(p^-)^*}}. \tag{73}$$

By using Fatou's Lemma, one obtains from (72) and (73),

$$\int_{\Omega} |T_t(u)|dx \leq (meas(\Omega))^{\frac{1}{(p^-)^*}} \left(meas(\Omega)(3+2t^{p^+})\right)^{\frac{1}{(p^-)^*}} \tag{74}$$

and

$$\int_{\Omega} |\nabla T_t(u)|dx \leq (meas(\Omega))^{\frac{1}{(p^-)^*}} \left(meas(\Omega)+t\|f\|_{L^1(\Omega)}\right)^{\frac{1}{(p^-)^*}}. \tag{75}$$

Let's put $A_t := \{x \in \partial\Omega : |T_t(u)| < t\}$ for all $t > 0$ and $C' = \partial\Omega \setminus \cup_{t>0} A_t$. One has

$$\begin{aligned} \sigma(C^{'}) = \frac{1}{t}\int_{C^{'}} |T_t(u)|dx &\leq \frac{1}{t}\int_{\partial\Omega} |T_t(u)|dx \\ &\leq \frac{C_2}{t}\|T_t(u)\|_{W^{1,1}(\Omega)} \\ &\leq \frac{C_2}{t}\|T_t(u)\|_{L^1(\Omega)} + \frac{C_2}{t}\|\nabla T_t(u)\|_{L^1(\Omega)}, \end{aligned}$$

where C_2 is a positive constant.
From (74) and (75), one obtains by letting $t \longrightarrow \infty$ that $\sigma(C') = 0$.
Let's define the function v on $\partial\Omega$ by $v(x) := T_t(u(x))$ if $x \in A_t$.
$\forall\ x \in \partial\Omega \setminus (C \cup C')$, $\exists t > 0$ such that $x \in A_t$ and one has

$$u_n(x) - v(x) = (u_n(x) - T_t(u_n(x))) + (T_t(u_n) - T_t(u(x))).$$

Since $x \in A_t$, one has $|T_t(u(x))| < t$, so $|T_t(u_n(x))| < t$. This implies that $|u_n(x)| < t$.
Thus, $u_n(x) - v(x) = (T_t(u_n(x)) - T_t(u(x))) \longrightarrow 0$ as $n \longrightarrow \infty$.
Therefore, u_n converges to v a.e. on $\partial\Omega$. As $u_n = \textit{constant}$ on $\partial\Omega$, one deduces that $v = \textit{constant}$. □

To complete the proof of existence of entropy solution it is necessary to show that the sequence $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ converges to $|u|^{p(x)-2}u$ in $L^1(\Omega)$. To do that, let us show that the sequence $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$. As u_n is a weak solution of problem (44) then, taking $\frac{1}{t}T_t(u_n - u_m)$ as test function in (45), one obtains

$$\int_{\Omega} \frac{1}{t}a(x,\nabla u_n).\nabla T_t(u_n-u_m)dx + \int_{\Omega} |u_n|^{p(x)-2}u_n\frac{1}{t}T_t(u_n-u_m)dx = \int_{\Omega}\frac{1}{t}f_nT_t(u_n-u_m)dx. \tag{76}$$

In a similar way for u_m with $\frac{1}{t}T_t(u_m - u_n)$ as test function, one obtains

$$\int_{\Omega} \frac{1}{t}a(x,\nabla u_m).\nabla T_t(u_m-u_n)dx + \int_{\Omega} |u_m|^{p(x)-2}u_m\frac{1}{t}T_t(u_m-u_n)dx = \int_{\Omega}\frac{1}{t}f_mT_t(u_m-u_n)dx. \tag{77}$$

By adding (76) and (77), follows

$$\begin{cases} \displaystyle\int_{\{|u_n-u_m|\leq t\}} \frac{1}{t}\left[a(x,\nabla u_n) - a(x,\nabla u_m)\right].(\nabla u_n - \nabla u_m)dx \\ \displaystyle + \int_{\Omega}\left(|u_n|^{p(x)-2}u_n - |u_m|^{p(x)-2}u_m\right)\frac{1}{t}T_t(u_n-u_m)dx \leq \int_{\Omega}|f_n - f_m|dx. \end{cases} \tag{78}$$

Letting $t \longrightarrow 0$ and as the first and the second terms in the left-hand side of inequality (78) are nonnegative, one gets

$$\int_{\Omega}\left||u_n|^{p(x)-2}u_n - |u_m|^{p(x)-2}u_m\right|dx \leq \int_{\Omega}|f_n - f_m|dx. \tag{79}$$

Since $(f_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$, by (79), one deduces that $\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$. As $L^1(\Omega)$ is a Banach space and $\xi \longmapsto |\xi|^{p(x)-2}\xi$ is continuous and strictly monotone then,

$$\left(|u_n|^{p(x)-2}u_n\right)_{n\in\mathbb{N}} \longrightarrow |u|^{p(x)-2}u \text{ in } L^1(\Omega). \tag{80}$$

One are now ready to apply the limit in (57).
The integral $\int_\Omega f_n(x)T_t(u_n-v)dx$ converges to $\int_\Omega f(x)T_t(u-v)dx$ since $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^1(\Omega)$ and $(T_t(u_n-v))_{n\in\mathbb{N}}$ converges weakly to $T_t(u-v)$ in $L^\infty(\Omega)$.
For the second term of (57), one has

$$\int_\Omega |u_n|^{p(x)-2}u_nT_t(u_n-v)dx = \int_\Omega \left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right)T_t(u_n-v)dx + \int_\Omega |v|^{p(x)-2}vT_t(u_n-v)dx.$$

The sequence $\left(\left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right)T_t(u_n-v)\right)_{n\in\mathbb{N}}$ is nonnegative and $\forall\ \xi \in \mathbb{R}^N$, $\xi \longmapsto |\xi|^{p(x)-2}$ is continuous, so one gets
$\left(\left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right)T_t(u_n-v)\right)_{n\in\mathbb{N}} \longrightarrow \left(|u|^{p(x)-2}u - |v|^{p(x)-2}v\right)T_t(u-v)$ a.e. in Ω. Then, by Fatou's lemma, it follows that

$$\int_\Omega \left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right)T_t(u_n-v)dx \leq \liminf_{n\longrightarrow\infty}\int_\Omega \left(|u_n|^{p(x)-2}u_n - |v|^{p(x)-2}v\right)T_t(u_n-v)dx. \tag{81}$$

Since $T_t(u_n-v)$ converges weakly to $T_t(u_n-v)$ in $L^\infty(\Omega)$ and

$$|(|v|^{p(x)-2}v)| = |v|^{p(x)-2}|v| = |v|^{p(x)-1} \in L^1(\Omega),$$

it follows that

$$\lim_{n\longrightarrow\infty}\int_\Omega |v|^{p(x)-2}vT_t(u_n-v)dx = \int_\Omega |v|^{p(x)-2}vT_t(u-v)dx. \tag{82}$$

Now, we write the first term in (57) as follow:

$$\int_\Omega a(x,\nabla u_n).\nabla T_t(u_n-v)dx = \int_{\{|u_n-v|\leq t\}} a(x,\nabla u_n).\nabla u_n dx - \int_{\{|u_n-v|\leq t\}} a(x,\nabla u_n).\nabla v dx. \tag{83}$$

Let's put $l = t + \|v\|_{L^\infty}$. Then,

$$\int_{\{|u_n-v|\leq t\}} a(x,\nabla u_n).\nabla v dx = \int_{\{|u_n-v|\leq t\}} a(x,\nabla T_l(u_n)).\nabla v dx. \tag{84}$$

Moreover, $(a(x,\nabla T_t(u_n)))_{n\in\mathbb{N}}$ is uniformly bounded in $\left(L^{p'}(\Omega)\right)^N$. From (H_3), relation (46) and Proposition 13, it converges weakly to $a(x,\nabla T_t(u))$ in $\left(L^{p'}(\Omega)\right)^N$. Therefore,

$$\lim_{n\longrightarrow\infty}\int_{\{|u_n-v|\leq t\}} a(x,\nabla T_l(u_n)).\nabla v dx = \int_{\{|u-v|\leq t\}} a(x,\nabla T_l(u)).\nabla v dx. \tag{85}$$

The sequence $(a(x,\nabla u_n).\nabla u_n)_{n\in\mathbb{N}}$ is nonnegative, then, according to Fatou's Lemma, one gets

$$\int_{\{|u-v|\}\leq t\}} a(x,\nabla u).\nabla u dx \leq \lim_{n\to\infty}\inf\int_{\{|u_n-v|\leq t\}} a(x,\nabla u_n).\nabla u_n dx. \tag{86}$$

From (81), (82), (85), (86), one gets

$$\int_\Omega a(x,\nabla u).\nabla T_t(u-v)dx + \int_\Omega |u|^{p(x)-2}uT_t(u-v)dx \leq \int_\Omega f(x)T_t(u-v)dx.$$

Thus, u is an entropy solution of problem (1).

References

[1] Benboubker M. B., Azroul E., Barbara A. *Quasilinear elliptic problems with nonstandard growth.* Electron. J. Differ. Equ. 2011, Paper No. **62**, 16 p. (2011).

[2] Boureanu M.M. *On some variable exponent problems with no-flux boundary condition.* In: Current Trends in Mathematical Analysis and Its Interdisciplinary Applications. Cham: Springer International Publishing, 2019. p. 253-285.

[3] Boureanu M.M., Udrea D. N. *Existence and multiplicity results for elliptic problems with $p(.)$-Growth conditions.* Nonlinear Anal., Real World Appl. 14, No. **4**, 1829-1844 (2013).

[4] Chen Y., Levine S., Rao R. *Variable exponent, linear growth functionals in image restoration.* SIAM J. Appl. Math. 66, No. **4**, 1383-1406 (2006).

[5] Diening L. *Theoretical and numerical results for electrorheological fluids.* Freiburg im Breisgau: Univ. Freiburg im Breisgau, Mathematische Fakult?t, 156 p. (2002).

[6] Fan X. *Solutions for p(x)B–Laplacian Dirichlet problems with singular coefficients.* J. Math. Anal. Appl. 312, No. **2**, 464-477 (2005).

[7] Halmos P. *Measure Theory.* University Series in Higher Mathematics. New York: D. Van Nostrand Co., Inc.; London: Macmillan and Co., Ltd. xii, 304 p. (1950).

[8] Kovacik O., Rakosnik J. *On spaces $L^{p(x)}$ and $W^{1,p(x)}$.* Czech. Math. J. 41(116), No. **4**, 592-618 (1991).

[9] Mihailescu M., Radulescu V. *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids.* Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 467, No. **2134**, 3033-3034 (2011).

[10] Ouaro S., Tchousso A. *Well-posedness result for a nonlinear elliptic problem involving variable exponent and Robin type boundary condition.* Afr. Diaspora J. Math. 11, No. **2**, 36-64 (2011).

[11] Ouaro S., Traoré S. *Weak and Entropy Solutions to Nonlinear Elliptic Problems with Variable Exponent.* J. Convex Anal. 16, No. **2**, 523-541 (2009).

[12] Rajagopal K., Ruzicka M. *Mathematical modeling of electrorheological materials.* Contin. Mech. Thermodyn. 13, No. **1**, 59-78 (2001).

[13] Ruzicka M. *Electrorheological fluids: modeling and mathematical theory.* Lecture Notes in Mathematics. 1748. Berlin: Springer. xvi, 161 p. (2000).

[14] Sanchon M., Urbano J. M. *Entropy solutions for the $p(x)$-Laplace equation.* Trans. Am. Math. Soc. 361, No. **12**, 6387-6405 (2009).

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Метою цієї роботи є дослідження ізотропної задачі з неоднорідною граничною умовою Діріхле та даними L^1 у просторі Соболєва зі змінним показником. Ми починаємо з показу існування та єдиності слабкого розв'язку, коли вихідний член обмежений. Нарешті, ми використовуємо проблемно-орієнтований підхід для доведення існування та єдиності ентропійного розв'язку, коли вихідний член є інтегровним.