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OPTIMAL CONTROL OF STOCHASTIC EQUATIONS OF POROUS ENVIRONMENTS

The technique of averaging of differential operators is applied to optimal control problems for stochastic equations describing porous media, which are parabolic equations with rapidly oscillating coefficients. It is proved that the optimal controls, trajectories, and minimum values of the quality criteria for the original problem converge, in the corresponding metrics, to the optimal controls, trajectories, and quality criteria of the averaged problems.

Key words and phrases: Wiener process, optimal control, weak convergence, averaging, compactness, reaction-diffusion, rapidly oscillating coefficients.

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine (Stanzhytskyi Oleksandr)
 National University «Kyiv aviation institute», Kyiv, Ukraine (Lastivka Ivan)
 Taras Shevchenko National University of Kyiv, Kyiv, Ukraine (Ryzhov Anton)
 Kruty Heroes Military Institute of Telecommunications and Information Technologies, Kyiv, Ukraine (Ryzhov Anton)
 e-mail: stanzhytskyi@knu.ua (Stanzhytskyi O.), ivan.lastivka@npp.kai.edu.ua (Lastivka I.), ryzhov@knu.ua (Ryzhov A.)

1 INTRODUCTION

It is well known that local characteristics of processes in micro-inhomogeneous (porous) media are characterized by diffusion matrices $a\left(\frac{x}{\varepsilon}\right)$, where $\varepsilon > 0$ is a small parameter. For small $\varepsilon > 0$, these matrices lead to rapidly oscillating coefficients, which significantly complicate the analysis. An effective approach in this situation is to replace the original parameters with their averaged counterparts [5]. Optimal control problems for parabolic equations with rapidly oscillating coefficients have been studied, for example, in [6, 7] and other works.

One of the first works to study nonlinear averaging problems for parabolic stochastic equations in bounded domains was [2]. In [4], a similar problem involving rapidly fluctuating coefficients in the stochastic term was considered. The averaging method was applied to the study of the long-term behavior of solutions to stochastic parabolic equations with rapidly fluctuating coefficients in the linear case in [12]. In [9], the averaging method was applied to the study of invariant measures of stochastic systems.

To the best of our knowledge, the averaging method has not previously been applied to optimal control problems for stochastic equations describing porous media. The present work is devoted to the study of such problems.

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Let $a(x)$, $x \in \mathbb{R}^d$, is a continuous, periodic, symmetric matrix on $[0, 1]^d$ that satisfies the uniform ellipticity condition: $\exists v > 0$ such that $\forall x \in \mathbb{R}^d, \forall \eta \in \mathbb{R}^d$,

$$\sum_{i,j=1}^d a_{ij}(x) \eta_i \eta_j \geq v |\eta|^2. \quad (1)$$

Here $|\cdot|$ denotes vector norm in $\mathbb{R}^d$. At the same time, due to continuity and periodicity, the coefficients $a_{ij}(x)$ of the matrix $a(x)$ are bounded on $\mathbb{R}^d$.

Let $f(x, y), \sigma(x, y) : \mathbb{R}^d \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be continuous in their arguments, periodic in x with period 1, and satisfy the Lipschitz condition with respect to y : $\exists L > 0$ such that

$$|f(x, y) - f(x, y_1)| + |\sigma(x, y) - \sigma(x, y_1)| \leq L|y - y_1|, \quad (2)$$

for arbitrary $x \in \mathbb{R}^d, y, y_1 \in \mathbb{R}^1$.

From the continuity and periodicity of f and σ it follows that they satisfy the linear growth condition: $\exists M > 0$ such that

$$|f(x, y)| + |\sigma(x, y)| \leq M(1 + |y|)$$

$\forall x \in \mathbb{R}^d, y \in \mathbb{R}^1$. In what follows, without loss of generality, we assume that $L = M$.

Let $D \subset \mathbb{R}^d$ be a bounded domain with a sufficiently regular boundary, for example, satisfying the Lyapunov condition. In the domain $D_T = (0, T) \times D$, we consider the following optimal control problem governed by the equation:

$$\begin{cases} dy_\varepsilon &= [\operatorname{div} (a(\frac{x}{\varepsilon}) \nabla y_\varepsilon) + f(\frac{x}{\varepsilon}, y_\varepsilon) + u(t, x)] dt + \sigma(\frac{x}{\varepsilon}, y_\varepsilon) dW(t), \\ y_\varepsilon(0, x) &= y^0(x), \\ y|_{\partial D} &= 0, \end{cases} \quad (3)$$

with the quality criterion

$$J_\varepsilon(u) = E \int_0^T \int_D y_\varepsilon^2(t, x) dx dt + E \int_0^T \int_D u^2(t, x) dx dt \rightarrow \inf. \quad (4)$$

Here $W(t)$ is a $L^2(D)$ -valued Q -Wiener process. The following stochastic process

$$u(t, x) = u(t, x, \omega) \in \mathbb{R}^1$$

we will take as the control parameter. In this case, we assume that there exists $\mathfrak{F}_t$ -adapted process $\varphi(t, x, \omega) \in \mathbb{R}^1$ such that $|u(t, x, \omega)| \leq \varphi(t, x, \omega)$ almost sure and

$$E \int_0^T \|\varphi(t)\|^2 dt < \infty. \quad (5)$$

More precise conditions for the problem will be provided in the main part of the paper.

Denote $\Pi = [0, 1]^d$. To the optimal control problem (3)-(4), we associate the following averaged problem:

$$\begin{cases} dy_0 &= [\operatorname{div} (a_0 \nabla y_0) + f_0(y_0) + u(t, x)] dt + \sigma(y_0) dW(t), \\ y_0(0, x) &= y^0(x), \\ y_0|_{\partial D} &= 0, \end{cases} \quad (6)$$

with the quality criterion

$$J_0(u) = E \int_0^T \int_D y_0^2(t, x) dx dt + E \int_0^T \int_D u^2(t, x) dx dt \rightarrow \inf. \quad (7)$$

Here a_0 is an averaged matrix in the sense of [5],

$$f_0(y) = \int_{\Pi} f(x, y) dx, \quad \sigma_0(y) = \int_{\Pi} \sigma(x, y) dx.$$

The main result of the paper is the establishment of the following variational relations. Let $(J_\varepsilon^*, u_\varepsilon^*, y_\varepsilon^*)$ be an optimal triple for the problem (3)-(4). Then

- 1) $J_\varepsilon^* \rightarrow J_0^*, \varepsilon \rightarrow 0$,
- 2) along a subsequence $u_\varepsilon^* \rightarrow u_0^*, y_\varepsilon^* \rightarrow y_0^*$ in $L^2(\Omega_T; H)$,

where (J_0^*, u_0^*, y_0^*) is the optimal triple for the problem (6)-(7). Here $H = L^2(D)$, $\Omega_T = \Omega \times [0, T]$, Ω is a sample space.

The paper is organized as follows. In Section 2, we introduce the necessary definitions and constructions and formulate the main results. Section 3 is devoted to the proof of several auxiliary statements. The main results are proved in Section 4.

2 NECESSARY DEFINITIONS AND MAIN RESULTS

In this work, we use the following functional spaces:

$$H = L^2(D), \quad V = H_0^1(D), \quad V' = H^{-1}(D).$$

Then $V \subset H \subset V'$ forms a Gelfand triple, and the embedding is compact. Let $\|\cdot\|$ be a norm in H , $(\cdot, \cdot)$ scalar product in this space, $\langle \cdot, \cdot \rangle$ – the spare between V and V' , i.e. $\langle z, v \rangle := z(v)$ for $z \in V'$, $v \in V$. Note that, by construction of V' we have that $\langle z, v \rangle = (z, v)$ for $z \in H$, $v \in V$.

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with normal filtration $(\mathfrak{F}_t)$, $t \in [0, T]$. Denote $D_t = D \times [0, T]$.

Let $\lambda_i > 0$ such that $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$, $\{l_i\}$ is an orthonormal basis in H , $l_i \in L^\infty(D)$ and $\sup_i \|l_i\|_{L^\infty(D)} < \infty$. We introduce the following H -valued stochastic process

$$W(t) := \sum_{i=1}^{\infty} \lambda_i l_i \beta_i(t), \quad t \geq 0,$$

which is called a Q -Wiener (kernel) process. Here $\beta_i(t)$ are standard, mutually independent one-dimensional Brownian motions. We will also assume that

- 1) $W(t)$ is $\mathfrak{F}_t$ -measurable, $t \in [0, T]$,
- 2) $W(t+h) - W(t)$ independent from the $\mathfrak{F}_t$ for $\forall t \geq 0, \forall h \geq 0$.

We also introduce the operator $Q \in L(H)$ such that Q is nonnegative and $Ql_i = \lambda_i^2 l_i$, $i \geq 1$. Denote

$$\lambda := \text{Tr} Q = \sum_{i=1}^{\infty} \lambda_i^2.$$

Let $L_2^0 = L^2\left(Q^{\frac{1}{2}}H; H\right)$ be the space of Hilbert–Schmidt operators equipped with the norm $\|\cdot\|_{L_2^0}$.

For the processes $\Phi : \Omega_T \rightarrow L\left(Q^{\frac{1}{2}}(H); H\right)$ such that $\Phi(t) \in \mathfrak{F}_t$ -measurable and

$$E \int_0^T \|\Phi \circ Q^{\frac{1}{2}}\|_{L_2^0}^2 dt < \infty.$$

following [10] we can define the stochastic integral

$$\int_0^t \Phi(s) dW(s) \in H$$

and

$$\int_0^t \Phi(s) dW(s) = \sum_{i=1}^{\infty} \lambda_i \int_0^t \Phi(s, \cdot) l_i(\cdot) d\beta_i(s),$$

and also

$$E \left\| \int_0^t \Phi(s) dW(s) \right\|^2 \leq \lambda \sup_i \|l_i\|_{L^\infty(D)}^2 \int_0^t E \|\Phi(s, \cdot)\|^2 ds.$$

Introduce an operator

$$A_\varepsilon y = \sum_{i,j=1}^d \partial_{x_i} \left(a_{ij} \left(\frac{x}{\varepsilon} \right) \partial_{x_j} y \right).$$

By the assumptions on the matrix $a(x)$, it is a coercive and continuous operator from V into V' .

Let $y^0 \in H$ be a $\mathfrak{F}_0$ -adapted, $u(t) \in H$ is a $\mathfrak{F}_t$ -measurable stochastic process.

Definition 2.1. $\mathfrak{F}_t$ -adapted stochastic process $y_\varepsilon(t) \in L^2(\Omega_T; H)$ is called a weak solution to the problem (3) on $[0, T]$ if for arbitrary $\varphi \in V$ an equality

$$\begin{aligned} (y_\varepsilon(t), \varphi) = & (y^0, \varphi) + \int_0^t \left(\langle A_\varepsilon y_\varepsilon(s), \varphi \rangle + (u(s), \varphi) + \left(f \left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s) \right), \varphi \right) \right) ds \\ & + \int_0^t \left(\varphi, \sigma \left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s) \right) \right) dW(s), \end{aligned} \quad (8)$$

holds for almost all $t \in [0, T]$ with probability 1.

It is well known [3] that under the conditions (1),(2) (and under more general conditions) such a solution exists and unique for the problem (3). In this case, the energy equality holds

$$\begin{aligned} \|y_\varepsilon(t)\|^2 = & \|y^0\|^2 + 2 \int_0^t \left(\langle A_\varepsilon y_\varepsilon(s), y_\varepsilon(s) \rangle + (u(s), y_\varepsilon(s)) + \left(f \left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s) \right), y_\varepsilon(s) \right) \right) ds \\ & + \int_0^t \left\| \sigma \left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s) \right) \right\|_{L_2^0}^2 ds + 2 \int_0^t \left(\sigma \left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s) \right) dW(s), y_\varepsilon(s) \right). \end{aligned} \quad (9)$$

In addition to the weak solution, we will also need the notion of a mild solution. Let us introduce the notion of a semigroup generated by the problem (3). Let $z_\varepsilon(t)$ be a strong solution to the problem

$$\begin{cases} \frac{dz_\varepsilon}{dt} = A_\varepsilon z_\varepsilon, \\ z(0) = y^0, \end{cases} \quad (10)$$

in V . Then $z_\varepsilon(t) = S_\varepsilon(t)y^0$, where $S(t)$ the corresponding semigroup of bounded operators, which can be extended to operators $S_\varepsilon(t) : H \rightarrow H$. In our case, it belongs to the class C_0 and is compact for $t > 0$.

If $K_\varepsilon(t, x, y)$ is the Green's function of the problem (10), then we have the integral representation

$$(S_\varepsilon(t)y^0)(x) = \int_D K_\varepsilon(t, x, y)y^0(y)dy. \quad (11)$$

If K is the Green's function of the problem (3) with the matrix $a(x)$, then

$$K_\varepsilon(x, y, t) = \varepsilon^{-d} K\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}; \frac{t}{\varepsilon^2}\right). \quad (12)$$

The Nash–Aronson estimates also hold; namely, there exists a positive constant $C_1(d, v)$ such that

$$\begin{aligned} 0 \leq K(x, y, t) &\leq C_1 t^{-\frac{d}{2}} \exp\left\{-\frac{v|x-y|^2}{8t}\right\}, \\ 0 \leq K_0(x, y, t) &\leq C_1 t^{-\frac{d}{2}} \exp\left\{-\frac{v|x-y|^2}{8t}\right\}. \end{aligned} \quad (13)$$

Here K_0 is the Green's function of the averaged problem with operator

$$A_0 y = \sum_{i,j=1}^d (a_0)_{ij} \partial_{x_i} \partial_{x_j} y.$$

From (11)-(13) it follows that there exists a constant $M = M(T)$ such that

$$\sup_{t \in [0, T]} \|S_\varepsilon(t)\|_{L(H; H)} \leq M. \quad (14)$$

Definition 2.2. $\mathfrak{F}_t$ -adapted process $y_\varepsilon(t) \in H$ is called the mild solution to the problem (3) on $[0, T]$, if the following equality holds

$$\begin{aligned} y_\varepsilon(t) = & S_\varepsilon(t)y^0 + \int_0^t S_\varepsilon(t-s)f\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s)\right) ds \\ & + \int_0^t S_\varepsilon(t-s)\sigma\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(s)\right) dW(s). \end{aligned}$$

Under conditions (1)-(2) from [10] it follows that the mild solution exist and unique; moreover, from [8] (under the same conditions) the existence of a weak solution implies the existence of a mild solution, and the two solutions coincide.

To solve the optimal control problems (3)-(4) and (6)-(7) we will need the notion of martingale solutions.

Definition 2.3. We say that problem (3) admits a weak martingale solution if there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, filtration $(\tilde{\mathcal{F}}_t)$, Q -Wiener $\tilde{\mathcal{F}}_t$ -adapted process $\tilde{W}(t)$, $\tilde{\mathcal{F}}_t$ -adapted process $\tilde{u}(t)$, $\tilde{y}^0 \in L^2(\tilde{\Omega}; H)$ such that $L(W, u, y^0) = L(\tilde{W}, \tilde{u}, \tilde{y}^0)$ and there exists a $\tilde{\mathcal{F}}_t$ -adapted process $\tilde{y}_\varepsilon \in L^2(\tilde{\Omega}_T; H)$ that $\tilde{y}_\varepsilon(t)$ satisfies (8) with W and u replaced by $\tilde{W}$ and $\tilde{u}$.

The notion of a mild martingale solution is defined analogously

If there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with a filtration $(\tilde{\mathcal{F}}_t)$ on which the processes $\tilde{u}(t)$ and $\tilde{\varphi}(t)$ can be defined such that they are $\tilde{\mathcal{F}}_t$ -adapted and $L(u, \varphi) = L(\tilde{u}, \tilde{\varphi})$, where $\{u\}$ is the set of admissible processes for the original problem, then the set $\{\tilde{u}\}$ is called *weakly admissible*.

The martingale solutions of (3) corresponding to weak admissible controls will be denoted by $\tilde{y}_\varepsilon(t, \tilde{u})$. The set of all weak admissible controls will be denoted by U_{ad} . Analogous notions are introduced for mild solutions.

Definition 2.4. We say that the pair $(\tilde{u}_\varepsilon^*(t), \tilde{y}_\varepsilon(t, \tilde{u}_\varepsilon^*))$ is a weak martingale solution to problem (3)–(4) if $\tilde{u}_\varepsilon^* \in U_{ad}$ and $\tilde{y}_\varepsilon(t, \tilde{u}_\varepsilon^*)$ is a solution to problem (3) in the sense of Definition 2.3 corresponding to the control $\tilde{u}_\varepsilon^*$, and

$$J_\varepsilon(\tilde{u}_\varepsilon^*) \leq J_\varepsilon(\tilde{u})$$

for arbitrary $\tilde{u} \in U_{ad}$.

We now state the main theorems of the paper

Theorem 2.1. Under conditions (1) and (2), problems (3)–(4) and (6)–(7) admit solutions.

Theorem 2.2. Under conditions (1) and (2) we have:

- 1) $J_\varepsilon^* \rightarrow J_0^*$;
- 2) a family of optimal controls $\{\tilde{u}_\varepsilon^*\}$ is strongly compact in $L^2(\tilde{\Omega}; H)$ and any weak limit $\tilde{u}$ in $L^2(\tilde{\Omega}; H)$ of a subsequence $\{\tilde{u}_{\varepsilon_n}^*\}$ is a weak martingale optimal control for the problem (6)–(7);
- 3) $\tilde{y}_{\varepsilon_n}^*(\cdot, \tilde{u}_{\varepsilon_n}^*) \rightarrow \tilde{y}(\cdot, \tilde{u})$ strongly in $L^2(\tilde{\Omega}_T; H)$ and $\tilde{y}(\cdot, \tilde{u})$ is the optimal trajectory of the averaged problem (6)–(7).

3 SOME AUXILIARY RESULTS

In this part of the paper, we present the necessary lemmas that play a key role in proving the main results. In the following two lemmas, we assume that the conditions (1), (2) and (5) are satisfied.

Lemma 3.1. Let $\{u_n\}$ is a sequence of admissible controls and $y_\varepsilon(t, u_n)$ are weak solutions corresponding to the controls u_n (ε is fixed). If $u_n \rightarrow u$ weakly in $L^2(\Omega_T; H)$, $n \rightarrow \infty$, then

$$y_\varepsilon(\cdot, u_n) \rightarrow y_\varepsilon(\cdot, u), n \rightarrow \infty, \text{ weakly in } L^2(\Omega_T; H).$$

Lemma 3.2. If $u_\varepsilon \rightarrow u_0$ weakly in $L^2(\Omega_T; H)$, $\varepsilon \rightarrow 0$, then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, sequences $\tilde{y}_{\varepsilon_n}$, $\tilde{u}_{\varepsilon_n}$, $\tilde{u}_0$ such that $L(\tilde{y}_{\varepsilon_n}, \tilde{u}_{\varepsilon_n}) = L(y_{\varepsilon_n}, u_{\varepsilon_n})$ i $L(u_0) = L(\tilde{u}_0)$ and

$$\tilde{y}_{\varepsilon_n}(\cdot, \tilde{u}_{\varepsilon_n}) \rightarrow \tilde{y}_0(\cdot, \tilde{u}_0) \text{ strongly in } L^2(\tilde{\Omega}_T; H).$$

Proof of the Lemma 3.1. Fix $\varepsilon > 0$ and let $y_\varepsilon(t, u_n)$ be a weak solution to (1). Note that, for each n such solution exists and it is unique. From the weak convergence we have the existence of the constant $C > 0$ such that

$$E \int_0^T \|u_n(t)\|^2 dt \leq C. \quad (15)$$

From (15), the energy equality (9), the conditions (1) and (2) it is easy to obtain the following inequality

$$\sup_{t \in [0, T]} E \|y_\varepsilon(t, u_n(t))\|^2 + \int_0^T E \|y_\varepsilon(t, u_n(t))\|_V^2 dt \leq C_1(y^0, T, C). \quad (16)$$

Therefore along a subsequence $u_n(\cdot) \rightarrow u(\cdot)$ weakly in $L^2(\Omega_T; H)$, $n \rightarrow \infty$, and $y_\varepsilon(\cdot, u_n(\cdot)) \rightarrow z_\varepsilon(\cdot)$ weakly in $L^2(\Omega_T; V)$, $n \rightarrow \infty$.

From (2) and (16) it is easy to obtain estimators

$$\begin{aligned} E \int_0^T \left\| f\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(t, u_n(t))\right) \right\|^2 dt &\leq C_2(C, C_1), \\ E \int_0^T \left\| \sigma\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(t, u_n(t))\right) \right\|_{L_2^\infty}^2 dt &\leq C_2(C, C_1), \end{aligned}$$

from which, along a subsequence, we have

$$\begin{aligned} f\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(\cdot, u_n)\right) &\rightarrow \Phi_\varepsilon(\cdot) \text{ weakly in } L^2(\Omega_T; H), n \rightarrow \infty, \\ \sigma\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(\cdot, u_n)\right) &\rightarrow \Psi_\varepsilon(\cdot) \text{ weakly in } L^2(\Omega_T; L_2^0), n \rightarrow \infty. \end{aligned}$$

Using standard arguments and the monotonicity conditions (see, e.g., [3, 8]), we conclude that $\Phi_\varepsilon = f\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(t, u)\right)$, $\Psi_\varepsilon = \sigma\left(\frac{\cdot}{\varepsilon}, y_\varepsilon(t, u)\right)$, and $z_\varepsilon = y_\varepsilon(t, u)$ is the solution to the problem (1) with the control u . By its uniqueness, it follows that all the convergences stated above hold for the entire sequence n . $\square$

Proof of the Lemma 3.2. We have

$$\begin{aligned} y_{\varepsilon_n}(t) &= S_{\varepsilon_n}(t)y^0 + \int_0^t S_{\varepsilon_n}(t-s)f\left(\frac{\cdot}{\varepsilon_n}, y_{\varepsilon_n}(s)\right) ds + \int_0^t S_{\varepsilon_n}(t-s)\sigma\left(\frac{\cdot}{\varepsilon_n}, y_{\varepsilon_n}(s)\right) dW(s) \\ &\quad + \int_0^t S_{\varepsilon_n}(t-s)u_{\varepsilon_n}(s)ds. \end{aligned}$$

It is obvious that for y_{ε_n} estimators of the type (16) are valid due to the uniform estimate

$$\sup_n E \int_0^T \|u_{\varepsilon_n}(t)\|^2 dt \leq C.$$

For arbitrary $R > 0$ we have

$$P \{ \|u_{\varepsilon_n}\|_{L^2(D_T)} > R \} \leq \frac{1}{R^2} \int_0^T E \|u_{\varepsilon_n}(t)\|^2 dt \rightarrow 0, R \rightarrow \infty,$$

uniformly on ε_n . Since the set of functions $\{\|u_{\varepsilon_n}\|_{L^2(D_T)} \leq R\}$ is weakly compact in $L^2(D_T)$, then the set of measures $L(u_{\varepsilon_n})$ is dense in $L^2(D_T)$, therefore, by the Prokhorov's theorem, it is weakly compact. By the generalized Skorokhod theorem (the Jakubowsky's theorem) we can define a

probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$, a filtration $\{\tilde{\mathfrak{F}}_t\}$, $\tilde{\mathfrak{F}}_t$ -adapted Wiener process $\tilde{W}(t)$, $\tilde{y}_{\varepsilon_n}(t)$, $\tilde{u}_{\varepsilon_n}(t)$, $\tilde{\varphi}(t)$, $\tilde{u}_0$ such that $L(y_{\varepsilon_n}, u_{\varepsilon_n}, \varphi(t), W(t)) = L(\tilde{y}_{\varepsilon_n}, \tilde{u}_{\varepsilon_n}, \tilde{\varphi}(t), \tilde{W}(t))$, $L(u_0) = L(\tilde{u}_0)$ i $\tilde{u}_{\varepsilon_n} \rightarrow \tilde{u}_0$, $\varepsilon_n \rightarrow 0$ weakly in $L^2(D_T)$ with probability 1 (along a subsequence). From the equality of distributions, we obtain that

$$\begin{aligned} \tilde{y}_{\varepsilon_n}(t) = & S_{\varepsilon_n}(t)\tilde{y}^0 + \int_0^t S_{\varepsilon_n}(t-s)f\left(\frac{\cdot}{\varepsilon_n}, y_{\varepsilon_n}(s)\right) ds + \int_0^t S_{\varepsilon_n}(t-s)\sigma\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_{\varepsilon_n}(s)\right) d\tilde{W}(s) \\ & + \int_0^t S_{\varepsilon_n}(t-s)\tilde{u}_{\varepsilon_n}(s)ds. \end{aligned} \quad (17)$$

Let $\tilde{y}_0(t, \tilde{u}_0)$ be a weak martingale solution to the problem (6). Then it is also a mild solution, and hence

$$\tilde{y}_0(t) = S_0(t)\tilde{y}^0 + \int_0^t S_0(t-s)f_0(\tilde{u}_0(s)) ds + \int_0^t S_0(t-s)\sigma(u_0(s)) d\tilde{W}(s) + \int_0^t S_0(t-s)\tilde{u}_0(s)ds. \quad (18)$$

From (17)-(18) we have

$$\begin{aligned} \tilde{E}\|\tilde{y}_{\varepsilon_n}(t) - \tilde{y}_0(t)\|^2 \leq & 4\left(\tilde{E}\|S_{\varepsilon_n}(t)\tilde{y}_0 - S_0(t)\tilde{y}_0\|^2 \right. \\ & + T \int_0^t E\left\| S_{\varepsilon_n}(t-s)f\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_{\varepsilon_n}(s)\right) - S_0(t-s)f_0(\tilde{y}_0(s)) \right\|^2 ds \\ & + \int_0^t E\left\| \sigma\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_{\varepsilon_n}(s)\right) - \sigma_0(\tilde{y}_0(s)) \right\|_{L_2^0}^2 ds \\ & \left. + \int_0^t E\|S_{\varepsilon_n}(t-s)\tilde{u}_{\varepsilon_n} - S_0(t-s)\tilde{u}_0(s)\|^2 ds \right) \\ = & I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (19)$$

Due to G -convergence of parabolic operators [11], the Lebesgue's dominated convergence theorem and the estimator (14)

$$I_1 \rightarrow 0, \varepsilon_n \rightarrow 0. \quad (20)$$

For the term I_2 , for some constant $C(T) > 0$, we have an estimator

$$\begin{aligned} I_2 \leq & C(T) \left(\int_0^t \tilde{E}\|S_{\varepsilon_n}(t-s)\|^2 \|\tilde{y}_{\varepsilon_n}(s) - \tilde{y}_0(s)\|^2 ds \right. \\ & \left. + \int_0^t \tilde{E}\left\| S_{\varepsilon_n}(t-s)f\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_0(s)\right) - S_0(t-s)f_0(\tilde{y}_0(s)) \right\|^2 ds \right). \end{aligned} \quad (21)$$

From the papers by [1] and [9] it follows with probability 1 that

$$f\left(\frac{x}{\varepsilon_n}, \tilde{y}_0(s, x, \omega)\right) \rightarrow f_0(\tilde{y}_0(s, x, \omega)) \text{ weakly in } L^2(D), \varepsilon_n \rightarrow 0,$$

for each $s \in [0, t]$. Then again due to G -convergence

$$\left\| S_{\varepsilon_n}(t-s)f\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_0(s)\right) - S_0(t-s)f_0(\tilde{y}_0(s)) \right\|^2 \rightarrow 0, \varepsilon_n \rightarrow 0,$$

with probability 1. Therefore

$$I_2 \rightarrow 0, \varepsilon_n \rightarrow 0, \quad (22)$$

by the linear growth conditions for f and f_0 and the Lebesgue's theorem.

Term I_3 can be estimated in a similar manner, taking into account the fact that

$$\left\| \sigma\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_{\varepsilon_n}(s)\right) - \sigma_0(\tilde{y}_0(s)) \right\|_{L_2^0}^2 \leq \lambda \sup_i \|l_i\|_{L^\infty(D)}^2 \left\| \sigma\left(\frac{\cdot}{\varepsilon_n}, \tilde{y}_{\varepsilon_n}(s)\right) - \sigma_0(\tilde{y}_0(s)) \right\|^2.$$

Thus

$$I_3 \rightarrow 0, \varepsilon_n \rightarrow 0. \quad (23)$$

For I_4 taking into account a G -convergence we have that

$$\|S_{\varepsilon_n}(t-s)\tilde{u}_{\varepsilon_n} - S_0(t-s)\tilde{u}_0\|^2 \rightarrow 0, \varepsilon_n \rightarrow 0, \quad (24)$$

for all t, s with probability 1.

Since $(u_{\varepsilon_n}, \varphi)$ and $(\tilde{u}_{\varepsilon_n}, \tilde{\varphi})$ are identically distributed, then for almost all (t, x, ω) we have $|\tilde{u}_{\varepsilon_n}(t, x, \omega)| \leq \tilde{\varphi}(t, x, \omega)$ and

$$\tilde{E} \int_0^T \|\tilde{\varphi}(t)\|^2 dt < \infty.$$

From (24) and Lebesgue's theorem it follows that

$$I_4 \rightarrow 0, \varepsilon_n \rightarrow 0. \quad (25)$$

Now from (19)-(23), (25) and Gronwall's lemma we obtain the Lemmas's statement. $\square$

4 PROOFS OF THE MAIN THEOREMS

Proof of the theorem 2.1. We provide the proof for the problem (3)-(4). Fix $\varepsilon > 0$. From the definition of the functional (4) follows that there exists a minimizing pair $(y_{\varepsilon_n}, u_{\varepsilon_n})$ such that

$$J_\varepsilon(u_{\varepsilon_n}) \rightarrow \inf_{u \in U_{ad}} J_\varepsilon = J_\varepsilon^*, n \rightarrow \infty.$$

For all n , we have that

$$|u_{\varepsilon_n}| \leq \varphi. \quad (26)$$

Therefore $\{u_{\varepsilon_n}\}$ weakly compact in $L^2(\Omega_T; H)$, hence along a subsequence, converges weakly to $u_\varepsilon \in L^2(\Omega_T; H)$. By Mazur's lemma and the inequality (26) follows, that $|u_\varepsilon(t, x, \omega)| \leq \varphi(t, x, \omega)$ for almost all (t, x, ω) . This gives us that u is the admissible control.

Without loss of generality, we may assume that the sequence u_{ε_n} itself converges weakly. Then, by Lemma 3.1 we obtain that $y_{\varepsilon_n}(t, u_{\varepsilon_n})$ also converges weakly in $L^2(\Omega_T; H)$ to $y_\varepsilon(t, u_\varepsilon)$.

By the weak lower semicontinuity of the norm, we have that

$$\begin{aligned} J_{\varepsilon_n}^* &= \lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}) \\ &= \liminf_{n \rightarrow \infty} \left(E \int_0^T \|y_{\varepsilon_n}(t, u_{\varepsilon_n})\|^2 dt + E \int_0^T \|u_{\varepsilon_n}(t)\|^2 dt \right) \\ &\geq E \int_0^T \|y_\varepsilon(t, u_\varepsilon)\|^2 dt + E \int_0^T \|u_\varepsilon(t)\|^2 dt. \end{aligned}$$

Thus the pair $(y_\varepsilon, u_\varepsilon)$ is the optimal process. For the problem (6)-(7) the proof is analogous. The theorem is proved. $\square$

Proof of the theorem 2.2. From the theorem 2.1 we have that the problems (3)-(4) and (6)-(7) have solutions, therefore, they also admit weak martingale solutions. Let us take as the underlying space $\tilde{\Omega} = [0, 1]$, $\tilde{P} = \lambda_{[0,1]}$ -Lebesgue measure on $[0, 1]$, $\tilde{\mathfrak{F}}$ is a Borel σ -algebra on $[0, 1]$. Denote by $(\tilde{y}_\varepsilon^*, \tilde{u}_\varepsilon^*)$ and $(y^*, \tilde{u}^*)$ optimal processes of the problems (3)-(4) and (6)-(7) respectively.

For each $\varepsilon > 0$ by the definition of an admissible control, we have that

$$\tilde{E} \int_0^T \|\tilde{u}_\varepsilon^*(t)\|^2 dt = \tilde{E} \int_0^T \int_D (\tilde{u}_\varepsilon^*(t)(t, x, \tilde{\omega}))^2 dx dt \leq \tilde{E} \int_0^T \int_D \tilde{\varphi}^2(t, x, \tilde{\omega}) dx dt < \infty.$$

Therefore the family $\{\tilde{u}_\varepsilon^*\}$ is weakly compact in $L^2(\tilde{\Omega}; H)$, hence, it contains a weakly convergent subsequence $\{\tilde{u}_{\varepsilon_n}^*\}$ such that

$$\tilde{u}_{\varepsilon_n}^* \rightarrow \tilde{u}_0, \varepsilon_n \rightarrow 0, \text{ weakly in } L^2(\tilde{\Omega}; H). \quad (27)$$

Then by the Lemma 3.2 there exists a probability space $(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathfrak{F}}}, \tilde{\tilde{P}})$, sequences $\tilde{\tilde{y}}_{\varepsilon_n}^*, \tilde{\tilde{u}}_{\varepsilon_n}^*, \tilde{\tilde{u}}_0^*$ such that $L(\tilde{\tilde{y}}_{\varepsilon_n}^*, \tilde{\tilde{u}}_{\varepsilon_n}^*) = L(\tilde{y}_{\varepsilon_n}^*, \tilde{u}_{\varepsilon_n}^*)$, $L(\tilde{\tilde{u}}_0) = L(\tilde{u}_0)$ and

$$\tilde{\tilde{y}}_{\varepsilon_n}^*(\cdot, \tilde{\tilde{u}}_{\varepsilon_n}^*) \rightarrow \tilde{\tilde{y}}_0(\cdot, \tilde{\tilde{u}}_0) \text{ strongly in } L^2(\tilde{\tilde{\Omega}}_T; H). \quad (28)$$

Note, however, that in the Skorokhod representation theorem, one can once again take $([0, 1], \mathfrak{B}([0, 1]), \lambda_{[0,1]})$ as the probability space $(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathfrak{F}}}, \tilde{\tilde{P}})$. From this $(\tilde{y}_{\varepsilon_n}^*, \tilde{u}_{\varepsilon_n}^*)$ and $(\tilde{\tilde{y}}_{\varepsilon_n}^*, \tilde{\tilde{u}}_{\varepsilon_n}^*)$ defined on the same probability space, hence, the pairs $(\tilde{\tilde{y}}_{\varepsilon_n}^*, \tilde{\tilde{u}}_{\varepsilon_n}^*)$ are optimal too. For simplicity of notation, we will omit the $\sim$ symbols from now on.

By weak lower semicontinuity we obtain

$$\liminf_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \geq E \int_0^T \|u_0(t)\|^2 dt. \quad (29)$$

From the other side, we have that

$$J_{\varepsilon_n}(u_{\varepsilon_n}^*) \leq J_{\varepsilon_n}(u_0),$$

therefore

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(E \int_0^T \|y_{\varepsilon_n}^*(t, u_{\varepsilon_n}^*)\|^2 dt + E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \right) \\ & \leq \liminf_{n \rightarrow \infty} \left(E \int_0^T \|y_{\varepsilon_n}(t, u_0)\|^2 dt + E \int_0^T \|u_0(t)\|^2 dt \right), \end{aligned}$$

which, in view of (28), yields the inequality

$$E \int_0^T \|y_0(t, u_0)\|^2 dt + \liminf_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \leq E \int_0^T \|y_0(t, u_0)\|^2 dt + E \int_0^T \|u_0(t)\|^2 dt.$$

Hence, we obtain that

$$\liminf_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \leq E \int_0^T \|u_0(t)\|^2 dt,$$

which, in view of (29), yields the equality

$$\liminf_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt = E \int_0^T \|u_0(t)\|^2 dt. \quad (30)$$

Hence, along a subsequence, which, without loss of generality, may be assumed to be the whole sequence,

$$E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \rightarrow E \int_0^T \|u_0(t)\|^2 dt,$$

which, together with (27), yields the equality

$$\lim_{k \rightarrow \infty} u_{\varepsilon_k}^* = u_0 \text{ in } L^2(\Omega_T; H).$$

The latter establishes the strong compactness of the family of optimal controls $\{u_\varepsilon^*\}$.

An obvious inequality is also given by

$$J_{\varepsilon_n}(u_{\varepsilon_n}^*) \leq J_{\varepsilon_n}(u^*),$$

where u^* is the optimal control for the averaged problem. Hence, by Lemma 3.2 we again obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}^*) &\leq \liminf_{n \rightarrow \infty} E \int_0^T \|y_{\varepsilon_n}(t, u^*)\|^2 dt + E \int_0^T \|u^*(t)\|^2 dt \\ &= E \int_0^T \|y^*(t, u^*)\|^2 dt + E \int_0^T \|u^*(t)\|^2 dt = J_0^*. \end{aligned} \quad (31)$$

Here, we have used the uniqueness of the solution to the averaged problem. On the other hand, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_{\varepsilon_n}^*) &= \lim_{n \rightarrow \infty} E \int_0^T \|y_{\varepsilon_n}^*(t, u_{\varepsilon_n}^*)\|^2 dt + \liminf_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \\ &= E \int_0^T \|y_0(t, u_0)\|^2 dt + E \int_0^T \|u_0(t)\|^2 dt = J_0(u_0). \end{aligned} \quad (32)$$

From (31)-(32) we have that the pair $(y_0(t, u_0), u_0)$ is optimal. Hence statements 2 and 3 are proved.

To prove statement 1, note that, by (28) and (30), we have

$$\begin{aligned} J &= \lim_{n \rightarrow \infty} J_{\varepsilon_n}^* = \lim_{n \rightarrow \infty} E \int_0^T \|y_{\varepsilon_n}^*(t, u_{\varepsilon_n}^*)\|^2 dt + \lim_{n \rightarrow \infty} E \int_0^T \|u_{\varepsilon_n}^*(t)\|^2 dt \\ &= E \int_0^T \|y_0(t, u_0)\|^2 dt + E \int_0^T \|u_0(t)\|^2 dt = J(u_0) = J_0^*. \end{aligned}$$

The theorem is proved. □

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До задач оптимального керування стохастичними рівняннями пористих середовищ, що є параболічними рівняннями зі швидко осцилюючими коефіцієнтами, застосовано техніку усереднення диференціальних операторів. Доведено, що оптимальні керування, траєкторії та мінімальні значення критеріїв якості точної задачі збігаються у відповідних метриках до оптимальних керувань, траєкторій та критеріїв якості усереднених задач.