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LINEAR $\mathcal{L}$-STRUCTURED MATRIX EQUATIONS

We obtained constructive necessary and sufficient conditions for the solvability and a scheme for constructing solutions of a linear matrix equation in the form of structured matrices. In particular case we obtained constructive necessary and suf- ficient conditions for the solvability and a scheme for constructing solutions of a linear matrix equation with an $\mathcal{L}$-structure. Particular instances of the $\mathcal{L}$-structure are magic squares, Hilbert, Hankel and Toeplitz matrices, Hermitian, symmetric and skew-symmetric matrices, as well as quaternions and biquaternions.

Key words and phrases: Linear matrix equation, structured matrices, Hilbert, Hankel and Toeplitz matrices, Hermitian, symmetric and skew-symmetric matrices, quaternions, biquaternions..

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1 Problem setup

We study the problem of constructing solutions [2, 6]

$$X \in \mathbb{R}^{\alpha\times\beta}$$

of a linear matrix equation

$$L(X) = F, \quad F \in \mathbb{R}^{\gamma\times\delta}, \tag{1}$$

where

$$L: \ \mathbb{R}^{\alpha\times\beta} \to \mathbb{R}^{\gamma\times\delta}$$

is a linear bounded matrix functional. Particular cases of the matrix algebraic equation (1) are the well-known Sylvester and Lyapunov equations [2, 3, 4]. In the article [14], the definition of several partial cases was introduced for a matrix algebraic equation (1) with an $\mathcal{L}$-structure

$$\mathcal{L}X = \mathcal{F}, \tag{2}$$

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which determines the linear relationship between the elements of the solution of the matrix algebraic equation. In particular, the $\mathcal{L}$-structure defines symmetric, skew-symmetric, diagonal matrices, as well as quaternions. Thus, we obtain the problem of finding solutions of a linear matrix equation (1) with an $\mathcal{L}$-structure defined by a linear bounded matrix functional

$$\mathcal{L}:\ \mathbb{R}^{\alpha\times\beta} \to \mathbb{R}^{\lambda\times\mu};$$

here $\mathcal{F} \in \mathbb{R}^{\lambda\times\mu}$ is a known matrix. Partial instances of the $\mathcal{L}$-structure are magic squares [15], Hilbert, Hankel and Toeplitz matrices [10], Hermitian [12], symmetric and skew-symmetric matrices, as well as quaternions and biquaternions [1, 13].

2 Conditions for the solvability of a linear matrix equation

We define the operator [6]

$$\mathcal{M}[A]:\ \mathbb{R}^{m\times n} \to \mathbb{R}^{m\cdot n},$$

as vectorization operator, which puts a matrix $A \in \mathbb{R}^{m\times n}$ into the column vector

$$\mathcal{B} := \mathcal{M}[A] \in \mathbb{R}^{m\cdot n},$$

composed of n columns of the matrix A and the inverse operator

$$\mathcal{M}^{-1}\Big[\mathcal{B}\Big]:\ \mathbb{R}^{m\cdot n} \to \mathbb{R}^{m\times n},$$

which puts the matrix A in correspondence to the vector $\mathcal{B} \in \mathbb{R}^{m\cdot n}$. Denote by [6]

$$\Big\{\Theta_j\Big\}_{j=1}^{\alpha\beta} \in \mathbb{R}^{\alpha\times\beta}$$

the natural basis of the space $\mathbb{R}^{\alpha\times\beta}$. We look for the general solution of the equation (1) in the form of a sum

$$X = \sum_{j=1}^{\alpha\beta} \Theta_j c_j, \quad c_j \in \mathbb{R}^1.$$

Denote the matrices

$$\Xi_j := L(\Theta_j) \in \mathbb{R}^{\gamma\times\delta}, \quad j = 1, 2,\ \dots,\ \alpha\beta.$$

Equation (1) is equivalent to the following equation

$$Q_0\, c = \mathcal{M}[F]$$

with respect to the vector $c \in \mathbb{R}^{\alpha\beta}$; here

$$Q_0 := \Big\{\ \mathcal{M}[\Xi_1]\ \ \mathcal{M}[\Xi_2]\ \dots\ \mathcal{M}[\Xi_{\alpha\beta}]\ \Big\} \in \mathbb{R}^{\gamma\delta\times\alpha\beta}.$$

Denote

$$P_{Q_0}:\ \mathbb{R}^{\alpha\beta} \to \mathbb{N}(Q_0), \quad P_{Q_0^*}:\ \mathbb{R}^{\gamma\delta} \to \mathbb{N}(Q_0^*)$$

the orthoprojectors corresponding matrices Q_0 and Q_0^*. Solvability conditions and the scheme for constructing solutions of a linear matrix equation 1 are determined by the following lemma Fredholm [6].

Lemma. *The matrix equation* (1) *is solvable iff*

$$P_{Q_0^*}\mathcal{M}[F]=0. \tag{3}$$

Under the condition (3) *and only under it, the equation* (1) *has an r-parametric family of solutions*

$$X=K[F]+V[c_r], \quad c_r\in\mathbb{R}^r,$$

where

$$K[F]:=\mathcal{M}^{-1}\Big\{Q_0^+\mathcal{M}[F]\Big\}, \quad V[c_r]:=\mathcal{M}^{-1}\Big[P_{Q_r}c_r\Big].$$

The matrix P_{Q_r} is formed from r linearly independent columns of the matrix-orthoprojector P_{Q_0}.

Linear operator

$$K[F]:\ \mathbb{R}^{\gamma\times\delta}\to\mathbb{R}^{\alpha\times\beta}$$

determines the partial solution of the inhomogeneous equation (1) and is an analogue of the Green's operator of the Cauchy problem for the system of ordinary differential equations [5, 7], at the same time, unlike Cauchy problems for a system of ordinary differential equations, the matrix equation (1) is not solvable for all right-hand sides. The matrix $V[c_r]$ determines the general solution of the homogeneous part of the equation (1).

3 Conditions for the solvability of a $\mathcal{L}$-structured matrix equations

Equation (2) is equivalent to the following equation

$$Q_1\,c=\mathcal{M}[\mathcal{F}]; \tag{4}$$

where

$$Q_1:=\Big\{\ \mathcal{M}[\Xi_1]\ \ \mathcal{M}[\Xi_2]\ ...\ \mathcal{M}[\Xi_{\alpha\beta}]\ \Big\}\in\mathbb{R}^{\lambda\mu\times\alpha\beta}.$$

Denote martix

$$\mathcal{Q}:=Q_1P_{Q_r}\in\mathbb{R}^{\lambda\mu\times r},$$

as well as matrices-orthoprojectors [5, 7]

$$P_{\mathcal{Q}}:\ \mathbb{R}^r\to\mathbb{N}(\mathcal{Q}), \quad P_{\mathcal{Q}^*}:\ \mathbb{R}^{\lambda\mu}\to\mathbb{N}(\mathcal{Q}^*).$$

Thus, we obtain an equation equivalent to the problem of finding solutions of a linear matrix equation (1) with a $\mathcal{L}$-structure (2):

$$\mathcal{Q}\,c=\mathcal{M}[\mathcal{F}]-Q_1Q_0^+\mathcal{M}[F],$$

is solvable iff [5]

$$P_{\mathcal{Q}^*}\Big\{\mathcal{M}[\mathcal{F}]-Q_1Q_0^+\mathcal{M}[F]\Big\}=0. \tag{5}$$

Under the conditions (3) and (5), the problem of finding solutions of the linear matrix equation (1) with $\mathcal{L}$-structure (2) has a ρ-parametric family of solutions

$$X=\mathcal{M}^{-1}\Big\{\mathcal{Q}^+\Big\{\mathcal{M}[\mathcal{F}]-Q_1Q_0^+\mathcal{M}[F]\Big\}\Big\}+\mathcal{M}^{-1}\Big[P_{\mathcal{Q}_\rho}c_\rho\Big], \quad c_\rho\in\mathbb{R}^\rho.$$

The matrix $P_{\mathcal{Q}_\rho}$ is formed from ρ linearly independent columns of the orthoprojector matrix $P_{\mathcal{Q}}$. The conditions for solvability and the structure of solutions of a matrix equation (1) are determined by the following theorem.

Theorem. *The problem of finding solutions of a linear matrix equation* (1) *with a $\mathcal{L}$-structure* (2) *under the conditions* (3) *and* (5) *has a ρ-parametric family of solutions*

$$X = G[F] + W[c_\rho], \quad c_\rho \in \mathbb{R}^\rho,$$

where

$$G[F] := \mathcal{M}^{-1}\Big\{\mathcal{Q}^+\Big\{\mathcal{M}[\mathcal{F}] - Q_1 Q_0^+ \mathcal{M}[F]\Big\}\Big\}, \quad W[c_\rho] := \mathcal{M}^{-1}\Big[P_{\mathcal{Q}_\rho} c_\rho\Big].$$

In the case of $P_{\mathcal{Q}^*} \neq 0$, the problem of finding solutions of a linear matrix equation (1) with a $\mathcal{L}$-structure (2) is solvable under the conditions (3) and (5). Therefore, we will call the case $P_{\mathcal{Q}^*} \neq 0$ critical, and the opposite case $P_{\mathcal{Q}^*} = 0$ noncritical.

Example 1. *The conditions of the proven theorem are fulfilled in the case of a linear matrix equation with a $\mathcal{L}$-structure*

$$LX = F, \quad \mathcal{L}X := MXN = \mathcal{F}. \tag{6}$$

Here

$$LX := \int_0^{2\pi} U(t)XV(t)dt, \quad V(t) := \begin{pmatrix} \sin t & \cos t & 0 & 0 \\ 0 & \sin t & \cos t & 0 \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

besides

$$U(t) := \frac{1}{\pi}\begin{pmatrix} \sin t & \cos t \end{pmatrix}, \quad F := \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$M := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The following matrix is key when studying the equation (6).

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

defines the orthoprojectors $P_{Q_0^*} = 0$ and

$$P_{Q_0} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_{Q_r} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

As a consequence of the equality $P_{Q_0^*} = 0$, the condition (3) is fulfilled, so the equation (6) has a two-parameter family of solutions

$$X = K[F] + V[c_r], \quad c_r \in \mathbb{R}^2;$$

here

$$K[F] := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V[c_r] := \begin{pmatrix} 1 & -c_1 & 1-c_2 \\ c_1 & 1+c_2 & 0 \end{pmatrix}, \quad c_r := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Matrix

$$\mathcal{Q} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ -2 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$$

defines the orthoprojectors $P_{\mathcal{Q}} = 0$ and

$$P_{\mathcal{Q}^*} = \frac{1}{15} \begin{pmatrix} 10 & 0 & 0 & 5 & -5 & 0 \\ 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 6 \\ 5 & 0 & 0 & 10 & 5 & 0 \\ -5 & 0 & 0 & 5 & 10 & 0 \\ 0 & 0 & 6 & 0 & 0 & 12 \end{pmatrix}.$$

The condition (5) is fulfilled, therefore the equation (6) with $\mathcal{L}$ - structure has a unique solution

$$X = G[F] = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In the noncritical case ($P_{\mathcal{Q}^*} = 0$), under the condition (3) the problem of finding solutions of the linear matrix equation (1) with the $\mathcal{L}$-structure (2) is solvable for any right parts.

Corollary. *In the noncritical case* ($P_{\mathcal{Q}^*} = 0$), *under the condition* (3) *the problem of finding solutions of the linear matrix equation* (1) *with the $\mathcal{L}$-structure* (2) *has a ρ-parametric family of solutions*

$$X = G[F] + W[c_\rho], \quad c_\rho \in \mathbb{R}^\rho,$$

where

$$G[F] := \mathcal{M}^{-1}\Big\{\mathcal{Q}^+\Big\{\mathcal{M}[\mathcal{F}] - Q_1 Q_0^+ \mathcal{M}[F]\Big\}\Big\}, \quad W[c_\rho] := \mathcal{M}^{-1}\Big[P_{\mathcal{Q}_\rho} c_\rho\Big].$$

Example 2. *The conditions of the proven corollary are fulfilled in the case of a linear matrix equation with a $\mathcal{L}$-structure*

$$LX = F, \quad \mathcal{L}X := MX + XN = \mathcal{F}. \tag{7}$$

Here

$$LX := \int_0^{2\pi} U(t)XV(t)dt, \quad V(t) := \begin{pmatrix} \sin t & \cos t & 0 & 0 \\ 0 & \sin t & \cos t & 0 \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

besides

$$U(t) := \frac{1}{\pi}\begin{pmatrix} \sin t & \cos t \end{pmatrix}, \quad F := \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

as well as

$$M := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad N := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The matrix Q_0, as well as the orthoprojectors $P_{Q_0^*} = 0$ and P_{Q_0} are given in example 1. As a consequence of the equality $P_{Q_0^*} = 0$, the condition (3) is fulfilled, so the equation (6) has a two-parameter family of solutions, given in example 1. Matrix

$$Q_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

is non-degenerate, therefore, according to the proven corollary, the linear matrix equation with the $\mathcal{L}$-structure (7) has a unique solution

$$X = G[F] = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In the partial case, the $\mathcal{L}$-structure defines real matrices of the form

$$A := \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}^1,$$

to which quaternions correspond

$$q = a + bi + cj + dk;$$

here i, j, k — imaginary units [13], [11, p. 91].

Example 3. *The conditions of the proved theorem are fulfilled in the case of the problem of finding a solution of a linear matrix equation*

$$LX = B, \quad \mathcal{L}X := MXN \tag{8}$$

in the form of a quaternion. Here

$$LX := R_1 X S_1 + R_2 X S_2,$$

besides

$$R_1 := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad R_2 := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad S_1 := \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_2 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 2 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 3 & 0 & -1 \end{pmatrix}.$$

The matrix is key when studying the equation (8)

$$Q_0 = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

defines orthoprojectors

$$P_{Q_0^*} = \frac{1}{2}\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1
\end{pmatrix}$$

and

$$P_{Q_0} = \frac{1}{9}\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -4 & 0 & 5 & 1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

besides

$$P_{Q_r} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The condition (3) is fulfilled, therefore the equation (8) has an eight-parameter family of solutions

$$X = K[F] + V[c_r], \quad c_r \in \mathbb{R}^8;$$

here

$$K[F] = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & -1 \\ 1 & 0 & 1 & 2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \quad V[c_r] = \begin{pmatrix} c_1 & c_3 & c_5 & c_7 \\ c_2 & c_4 & c_6 & c_8 \\ -c_1 & -c_3 & -c_5 & -c_7 \\ -c_2 & -c_4 & -c_6 & -c_8 \end{pmatrix}, \quad c_r := \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_8 \end{pmatrix}.$$

Matrix

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

defines the orthoprojectors $P_{Q_1} = 0$ and

$$P_{Q_1^*} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The condition (5) is fulfilled, so the equation (8) with the $\mathcal{L}$-structure has a unique solution in the form of a quaternion

$$X = G[F] = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix}.$$

The scheme proposed in the article for studying the solvability conditions and constructing solutions of the matrix equation with $\mathcal{L}$-structure can be transferred to linear matrix equations of a more general form [6], as well as to nonlinear matrix equations with $\mathcal{L}$-structure [8, 9], in particular, to nonlinear Riccati matrix equations [4].

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Ми отримали конструктивні необхідні та достатні умови розв'язності та схему побудови розв'язків лінійного матричного рівняння у вигляді структурованих матриць. У окремому випадку ми отримали конструктивні необхідні та достатні умови розв'язності та схему побудови розв'язків лінійного матричного рівняння зі структурою $\mathcal{L}$. Окремими випадками структури $\mathcal{L}$ є магічні квадрати, матриці Гільберта, Ганкеля та Тепліца, ермітові, симетричні та кососиметричні матриці, а також кватерніони та бікватерніони.